D=10, N=1 Supergravity and Second Order String Corrections

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Abstract

We study the Low Energy Limit of string theory in the form of string corrected Supergravity. We do this in the non minimal case. That is we solve the case of string corrected, D=10, N=1 Supergravity at second order in the string slope parameter. We obtain the second order H sector tensors, as well as the related torsions and curvatures. We also find the supercurrent supertensor at second order.

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Contents

1 Introduction

It is established tradition that theories of extended objects have eclipsed supergravity as a focus of attention in the search for a consistent theory of all known interactions. However in the past, common ground for both theories was found, (see references [1] through [5], and references therein.) It is believed that D=10, N=1 Supergravity is in fact the low energy limit to string theory, [1]. The first order or minimal Lorentz Chern-Simmons Form string modifications or deformations to D=10, N=1 Supergravity, that was manifestly supersymmetric, was first completed in [4]. That is, the authors succeeded in closing the related Bianchi identities to first order in the string slope parameter, obtaining the minimal string corrections. The completion of the task at second order or in the non-minimal case has remained unsolved until now. It was not known how to close the Bianchi identities at second order. Also the relevance of such a lengthy task was not fully realized until recently when the first order case was revisited. The results of [4] have recently been strongly vindicated, ([2] and references therein). Hence there has been a renewed focus of interest in the non-minimal case, [2].

In view of this new interest in string corrected D=10, N=1 Supergravity, we therefore revisit an outstanding problem concerning the case at second order in the perturbative expansion. Some years ago a program to incorporate string corrections into the supergravity equations of motion which succeeded in maintaining manifest supersymmetry was developed, [1], [2], [3], [4]. Recently the bosonic equations of motion for D=10, N=1 supergravity fields at superspace and component levels have been derived and have been

shown to be derivable from a lagrangian, [2]. This was done to first order perturbatively in the string slope parameter. A major problem continued to be that of obtaining the solution of the Bianchi identities at second order. It was suggested, [4], that the second order case would contain many interesting results, and that it would follow by merely obtaining perturbative expansions of the super tensors, H_{abc}, L_{abc}, A_{abc}. However obtaining the second order result did not come so easily. (Some authors maintained a different approach to that in [2], see for example [5]).

Closing the Bianchi identities at second order first ground to a halt upon encountering a non-solvable term in the H sector Bianchi identities. Many approaches such as null spaces and other ideas were tried in order to overcoming this problem.

It was suggested that one aspect of finding a second order solution would require the so called X tensor, [1], [7]. Finding the X tensor would form part of the solution, and finding it through direct calculations proved difficult. However this author was aware for a long time that a particular Ansatz did in fact allow for closure in the H sector,(equation (69)). However it was required to find a consistent set of torsions and curvatures also. Eventually, an equation was derived which allowed for a solution up to a curvature, equation (39). The remaining curvature looked intractable until yet a further relation was found, (equation (124)). Finally a condition to be imposed on the curvature ${R_{ab\alpha}}^{\beta}$ (equation 129), which was not evident in referenc [4], was implemented, (129).

Hence in this work we propose a candidate for the X tensor and show that it allows for a solution in the H sector. Furthermore we solve the related torsion identities. We also find ${R^{(2)}}_{\alpha\beta de}$, ${R^{(2)}}_{\alpha bde}$. We find ${{R^{(1)}}_{ab\gamma}}^{\delta}$ and note that it appears to be set to zero in

[2] and [4]. Finally we find ${A^{(2)}}_{abc}$, the super current super tensor at second order.

The perturbative approach of Gates and collaborators is well documented and discussed in the literature, and we will not recount it here. For a recent review and for an up to date commentary see [2], and references therein. For a discussion of Bianchi identities in general see [6]. The crucial role of the Chern-Simmons form is discussed in [8] and [4]. Our starting point will be the Bianchi identities as listed in [4]. The sigma matrix identities and symmetries are recorded in [3].

These geometrical methods nowadays are known as deformations, [2], and the constraints have sometimes been referred to in the past as Beta Function Favored constraints, (βFF constraints).

2 The Bianchi Identities

Bianchi Identites in the presence of constraints can give information about dynamics. Perhaps the best known example in physics is the case of the Einstein Field Equations. It is well known that these field equations can be derived from an action, the Einstein Hilbert action. However, as is also well known, the standard route towards their derivation is through the contraction of the Bianchi identities in classical Riemannian space. An example of a physical constraint is that of the conservation of the energy momentum tensor. Hence although the Bianchi identities are in fact just identities, if a constraint is imposed on one or more fields, this in turn will generate constraints on other fields related through these identities.

In our case we consider in a slightly analogous way, those Bianchi identities derivable in superspace in conjunction with a set of constraints. For details of the early work on this see for example [1] and references therein. Following the notation of [6] we define torsions $T_{AB}{}^{C}$ and curvatures $R_{AB}{}^{CD}$ as follows

$$[\nabla_A, \nabla_B\} = T_{AB}{}^{C} + \frac{1}{2} R_{ABd}{}^{e} M_e{}^{d} \tag{1}$$

The Bianchi identities are given by

$$[[\nabla_{[A}, \nabla_B\}, \nabla_{C)}\} = 0 \tag{2}$$

For further details see appendix [II]. Firstly we consider the set of Q and G Bianchi identities is listed in ref. [4]. Q_{ABD} is the Super Lorentz Chern Simmons Form. G_{ABD} is the supergravity field strength for D=10, N=1 Supergravity. H_{ABD} will be the modified supergravity field strength for D=10, N=1 Supergravity

The Bianchi identities satisfied by the Lorentz Chern Simmons form, Q_{ABD}, are [4]

$$\frac{1}{6}\nabla_{(\alpha|} Q_{|\beta\gamma\delta)} - \ \frac{1}{4} T_{(\alpha\beta|}{}^{E} Q_{E|\gamma\delta)} = \ \frac{1}{4} R_{(\alpha\beta|ef} R_{|\gamma\delta)}{}^{ef} \tag{3}$$

$$\frac{1}{2}\nabla_{(\alpha|} Q_{|\beta\gamma)d} - \nabla_d Q_{\alpha\beta\gamma} \ - \ \frac{1}{2} T_{(\alpha\beta|}{}^{E} Q_{E|\gamma)d} \ + \ \frac{1}{2} T_{d(\alpha|}{}^{E} Q_{E|\beta\gamma)} \ = \ R_{(\alpha\beta|ef} R_{|\gamma)d}{}^{ef} \tag{4}$$

$$\nabla_{(\alpha|} Q_{\beta)cd} \ + \ \nabla_{[c|} Q_{|d]\alpha\beta} \ - \ T_{\alpha\beta}{}^{E} Q_{Ecd} \ - \ T_{cd}{}^{E} Q_{E\alpha\beta} \ - \ T_{(\alpha|[c|}{}^{E} Q_{E|d]|\beta)} \ =$$

$$[2R_{\alpha\beta ef}R_{cd}{}^{ef} \ + \ R_{(\alpha|[c|}{}^{ef}R_{|d]|\beta)ef}] \tag{5}$$

$$\nabla_\alpha Q_{bcd} - \frac{1}{2}\nabla_{[b|}Q_{\alpha||cd]} - \frac{1}{2}T_{\alpha[b|}{}^{E}Q_{E|d]\alpha} + \frac{1}{2}T_{[bc|}{}^{E}Q_{E|d]\alpha} = R_{\alpha[b|}{}^{ef}R_{|cd]ef} \tag{6}$$

$$+\frac{1}{6}\nabla_{[a|}Q_{|bcd]} - \frac{1}{4}T_{[ab|}{}^{E}Q_{E|d]\alpha} + \frac{1}{2}T_{[bc|}{}^{E}Q_{E|cd]} = \frac{1}{4}R_{[ab|}{}^{ef}R_{|cd]ef} \tag{7}$$

Those satisfied by the supertensor, G_{ABC}, are, [4],

$$\frac{1}{6}\nabla_{(\alpha|}G_{|\beta\gamma\delta)} - \ \frac{1}{4}T_{(\alpha\beta|}{}^{E}G_{E|\gamma\delta)} = 0 \tag{8}$$

$$\frac{1}{2}\nabla_{(\alpha|}G_{|\beta\gamma)d} - \nabla_d G_{\alpha\beta\gamma} \ - \ \frac{1}{2}T_{(\alpha\beta|}{}^{E}G_{E|\gamma)d} \ + \ \frac{1}{2}T_{d(\alpha|}{}^{E}G_{E|\beta\gamma)} \ = \ 0 \tag{9}$$

$$\nabla_{(\alpha|}G_{\beta)cd} \ + \ \nabla_{[c|}G_{|d]\alpha\beta} \ - \ T_{\alpha\beta}{}^{E}G_{Ecd} \ - \ T_{cd}{}^{E}G_{E\alpha\beta} \ - \ T_{(\alpha|[c|}{}^{E}G_{E|d]|\beta)} \ = 0 \tag{10}$$

$$\nabla_\alpha G_{bcd} - \frac{1}{2}\nabla_{[b|}G_{\alpha||cd]} - \frac{1}{2}T_{\alpha[b|}{}^{E}G_{E|d]\alpha} + \frac{1}{2}T_{[bc|}{}^{E}G_{E|d]\alpha} = 0 \tag{11}$$

$$+\frac{1}{6}\nabla_{[a|}G_{|bcd]} - \frac{1}{4}T_{[ab|}{}^{E}G_{E|d]\alpha} + \frac{1}{2}T_{[bc|}{}^{E}G_{E|cd]} = 0 \tag{12}$$

The Q and G tensors are related the modified field strength tensor, the H tensor, in superspace as follows

$$G_{ADG} \ = H_{ADG} \ + \ \gamma Q_{ADG} + \beta Y_{ADG} \tag{13}$$

Y_{ADG} is the Yang Mills Superform, and γ is proportional to the string slope parameter. That is, the action for massless fields of heterotic or type I superstrings may be expanded, with β set to zero as follows, [1],

$$S_{eff} \ = \ \frac{1}{\kappa^2}\int d^{10}xe^{(-1)}[L_{(0)} \ + \ \sum_{n=1}^{n=\infty}(\gamma')^n L_{(n)}] \tag{14}$$

Thus we can arrive at the low energy string corrected effective action.

We now consider the Bianchi identities satisfied by the tensor H_{ABC} which are as followes

$$\frac{1}{6}\nabla_{(\alpha|}H_{|\beta\gamma\delta)} - \ \frac{1}{4}T_{(\alpha\beta|}{}^{E}H_{E|\gamma\delta)} = \ (-\frac{\gamma}{4})R_{(\alpha\beta|ef}R_{|\gamma\delta)}{}^{ef} \tag{15}$$

$$\frac{1}{2}\nabla_{(\alpha|}H_{|\beta\gamma)d} - \nabla_d H_{\alpha\beta\gamma} \ - \ \frac{1}{2}T_{(\alpha\beta|}{}^{E}H_{E|\gamma)d} \ + \ \frac{1}{2}T_{d(\alpha|}{}^{E}H_{E|\beta\gamma)} \ = \ (-\gamma)R_{(\alpha\beta|ef}R_{|\gamma)d}{}^{ef} \tag{16}$$

$$\nabla_{(\alpha|}H_{\beta)cd} \ + \ \nabla_{[c|}H_{|d]\alpha\beta} \ - \ T_{\alpha\beta}{}^{E}H_{Ecd} \ - \ T_{cd}{}^{E}H_{E\alpha\beta} \ - \ T_{(\alpha|[c|}{}^{E}H_{E|d]|\beta)} \ = \\ -\gamma[2R_{\alpha\beta ef}R_{cd}{}^{ef} \ + \ R_{(\alpha|[c|}{}^{ef}R_{|d]|\beta)ef}] \tag{17}$$

$$\nabla_\alpha H_{bcd} - \frac{1}{2}\nabla_{[b|} H_{\alpha||cd]} - \frac{1}{2} T_{\alpha[b|}{}^E H_{E|d]\alpha} + \frac{1}{2} T_{[bc|}{}^E H_{E|d]\alpha} = -\gamma R_{\alpha[b|}{}^{ef} R_{|cd]ef} \tag{18}$$

$$+\frac{1}{6}\nabla_{[a|} H_{|bcd]} - \frac{1}{4} T_{[ab|}{}^E H_{E|d]\alpha} + \frac{1}{2} T_{[bc|}{}^E H_{E|cd]} = -\frac{1}{4}\gamma R_{[ab|}{}^{ef} R_{|cd]ef} \tag{19}$$

Within the framework of the Bianchi identities we have a perturbative prescription that will allow us to incorporate string corrections into the theory, and maintain it manifestly supersymmetric. We first solve the identities satisfied by the H tensor. To find them we use the Bianchi identities as satisfied by the G tensor and the Lorentz Chern Simmons form as listed in [4]. We then combine them to get the H sector Bianchi Identities. We also have the torsions

$$T_{(\alpha\beta|}{}^\lambda T_{|\gamma)\lambda}{}^d \;-\; T_{(\alpha\beta|}{}^g T_{|\gamma)g}{}^d \;-\; \nabla_{(\alpha|} T_{\beta\gamma)}{}^d \;=\; 0 \tag{20}$$

$$T_{(\alpha\beta|}{}^\lambda T_{|\gamma)\lambda}{}^\delta \;-\; T_{(\alpha\beta|}{}^g T_{|\gamma)g}{}^\delta \;-\; \nabla_{(\alpha|} T_{|\beta\gamma)}{}^\delta - \;\frac{1}{4} R_{(\alpha\beta|de} \sigma^{de}{}_{|\gamma)}{}^\delta \;=\; 0 \tag{21}$$

And we also have the following curvature

$$T_{(\alpha\beta|}{}^\lambda R_{|\gamma)\lambda de} \;-\; T_{(\alpha\beta|}{}^g R_{|\gamma)gde} \;-\; \nabla_{(\alpha|} R_{\beta\gamma)de} \;=\; 0 \tag{22}$$

There are many other identities but these are the principle ones needed for this work. In fact we only need to solve the first three H sector identities to find a full solution for the H sector tensors in terms of curvatures and torsions.

The first order solutions were first given in [4]. They were recently recalculated [2]. Using conventional constraints as input, Bellucci, Gates and Depireux derived the first order results, as given in [4]. The conventional constraints, (where here we use a slightly different convention) are, [2],

$$
\begin{aligned}
i\sigma_a{}^{\alpha\beta}T_{\alpha\beta}{}^b = 16\delta_a^b \quad &, i\sigma_c{}^{\alpha\beta}T_{\alpha b}{}^c = 0 \\
i\sigma_{abcde}{}^{\alpha\beta}T_{\alpha\beta}{}^e = 0, \qquad & T_{\alpha[de]} = 0 \\
T_{deb} = \frac{1}{8}\sigma_{de\alpha}{}^{\beta}T_{\beta b}{}^{\alpha} - & \frac{i}{16}R_{\alpha\beta\ de}
\end{aligned}
\tag{23}
$$

An important input at first order is what is taken to be the supercurrent supertensor A_{abc}.

The choice with β set to zero of

$$
A_{abc} = -i\gamma\sigma_{abc\epsilon\tau}T_{kp}{}^{\epsilon}T^{kp\tau} \tag{24}
$$

was made to put the theory on shell, [2].

In this work we also find A_{abc} at second order. We will list the main results at first order found in [2] and [4]. This is because we continually refer to them. We also wish to

establish out notation, and because we need these results in terms of the H tensor, not the G tensor as given in [4]. We list only those which we require, and we set $\beta = 0$. (ie we include no matter fields). We will note however that we will later have to modify the constraint on $T_{\alpha\beta}{}^{g}$.

We have at first order

$$H_{\alpha\beta\gamma} = 0 \qquad H_{\alpha\beta d} = +\frac{i}{2}\sigma_{d\alpha\beta} + 4i\gamma\sigma^{g}{}_{\alpha\beta}H_{\gamma}{}^{ef}H_{d}{}^{ef}$$
$$H_{\alpha bd} = +2i\gamma[-\sigma_{[b|\alpha\beta}T_{ef}{}^{\beta}G_{|d]}{}^{ef} - 2\sigma_{e\alpha\beta}T_{f[b|}{}^{\beta}G_{|d]}{}^{ef}] \tag{25}$$

$$T_{\alpha\beta}{}^{g} = i\sigma_{\alpha\beta}{}^{g}, \quad T_{\alpha b}{}^{g} = 0, \quad T_{abc} = -2L_{abc},$$
$$T_{\alpha\beta}{}^{\gamma} = -[\delta_{(\alpha|}{}^{\gamma}\delta_{|\beta)}{}^{\delta} + \sigma^{g}{}_{\alpha\beta}\sigma_{g}{}^{\gamma\delta}]\chi_{\delta}$$
$$T_{\alpha b}{}^{\gamma} = \frac{1}{48}\sigma_{b\alpha\phi}\sigma^{pqr\phi\gamma}A_{pqr} \tag{26}$$

$$R_{\alpha\beta de} = -2i\sigma^{g}{}_{\alpha\beta}\Pi_{gde} + \frac{i}{24}\sigma^{pqref}{}_{\alpha\beta}A_{pqr}$$
$$R_{\alpha bde} = -i\sigma_{[d|\alpha\phi}T_{b|e]}{}^{f} + i\gamma\sigma_{[d|\alpha}{}_{\ \phi}T_{kl}{}^{\phi}R^{kl}{}_{|de]} \tag{27}$$

Where

$$\Pi^{(1)}{}_{g}{}^{ef} = L^{(1)}{}_{g}{}^{ef} - \frac{1}{8}A^{(1)}{}_{g}{}^{ef} \tag{28}$$

$$L_{abc} = H_{abc} + \gamma[(R^{ef}{}_{|ab]} + R_{|ab]}{}^{ef} - \frac{8}{3}H_d{}^{ef}{}_{[a|}H^{df}{}_{|b]})H_{|c]ef}] \tag{29}$$

$$\begin{aligned}
\nabla_\alpha L_{bcd} &= \frac{i}{4}\sigma_{[b|\alpha\beta}[T_{|cd]}{}^{\beta} + 4\gamma T_{kl}{}^{\beta}R^{kl}{}_{|cd]} \\
\nabla_\gamma T^{(0)}{}_{ef}{}^{\delta} &= [-\frac{1}{4}\sigma^{mn}{}_{\gamma}{}^{\delta}R_{efmn} \;+\; T^{(0)}{}_{ef}{}^{\lambda}T^{(0)}{}_{\gamma\lambda}{}^{\delta}] \\
&+\frac{1}{48}[2H_{efg}\sigma^{g}{}_{\gamma}{}^{\phi}\sigma^{dkl}{}_{\phi}{}^{\delta} - \sigma_{[e|\gamma}{}^{\phi}\sigma^{dkl}{}_{\phi}{}^{\delta}\nabla_{|f]}]A_{dkl}
\end{aligned} \tag{30}$$

Many avenues such as null spaces were tried in order to solve the problem of finding a solution at second order, but without success. However it was suggested that a generalization of the torsion $T_{\alpha\beta}{}^{g}$ would be necessary in order to proceed to second order, [2],[7]. The job at hand therefore is to find the form of this generalization, known as the X tensor. However there were still many obstacles to be overcome in order to obtain a complete and consistent set of solutions.

In this paper we propose a candidate for the X tensor. We show that this candidate solves the problem of closure in the H sector Bianchi Identities. It also solves the second order torsion. We therefore find the second order torsions $T^{(2)}{}_{\alpha\beta}{}^{g}$, $T^{(2)}{}_{\alpha\beta}{}^{\lambda}$, $T^{(2)}{}_{\alpha b}{}^{g}$ and $T^{(2)}{}_{\alpha b}{}^{\gamma}$ and also the curvatures $R^{(2)}{}_{\alpha bde}$, and $R^{(2)}{}_{\alpha\beta de}$. In order to do this we required several insights and key results which are derived in appendices.

We check our results by showing mutual consistency. At this stage we will draw attention to our simple second order notation. The superscript in brackets refers to the second order quantities. Hence we have as follows for example

$$R_{\alpha\beta de} = R^{(0)}{}_{\alpha\beta de} + R^{(1)}{}_{\alpha\beta de} + R^{(2)}{}_{\alpha\beta de} + ... \tag{31}$$

Where $R^{(0)}{}_{\alpha\beta de}$ and $R^{(1)}{}_{\alpha\beta de}$ are listed in equations (27). To begin with, $H_{\beta\gamma\delta}$ is set to zero as in [2] and [4]. We have not seen that it is required to be other than zero to close the Bianchi identities. We have seen that if it is non zero the H sector Bianchi Identities fail to close.

3 The X Tensor

In reference [4] the conventional constraint $T^{g}{}_{\alpha\beta} \;=\; i\sigma^{g}{}_{\alpha\beta}$ was imposed to all orders. This led to failure to close the Bianchi identities at second order. In this case that constraint is modified. From the conventional constraints listed in [2] the most general form of the zero dimensional torsion is

$$T_{\alpha\beta}{}^{g} \;=\; i\sigma^{g}{}_{\alpha\beta} \;+\; \sigma^{pqref}{}_{\alpha\beta}X_{pqref}{}^{g} \tag{32}$$

Here we absorb the coefficient $\frac{i}{5!}$ used in ref. [2] into the X tensor.

Earlier because of the existence of an apparently intractable term which arose in the H sector Bianchi identities, closure could not be obtained with $T_{\alpha\beta}{}^{g} = i\sigma^{g}{}_{a\beta}$. At the time the problem term could not be incorporated into the torsion $T^{(2)}{}_{\alpha\beta}{}^{\lambda}$, which would have allowed for a solution. In the following we see that the X tensor must contribute to second order. Consider the Bianchi identity at dimension one half, equation (20). If X is zero,

then using the constraints in [4], reduces this equation to

$$T^{(2)}{}_{(\alpha\beta|}{}^{\lambda}\sigma_{|\gamma)\lambda}{}^{d} \;=\; \sigma_{(\alpha\beta|}{}^{g}T^{(2)}{}_{|\gamma)g}{}^{d} \tag{33}$$

Therefore $T^{(2)}{}_{\alpha\beta}{}^{\lambda}$ must have a similar sigma matrix structure to the RHS of (33). We know from the H sector Bianchi identities that $H^{(2)}{}_{g\gamma d}$ satisfies an equation of the form

$$\begin{aligned}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} \quad &= \\ \sigma^{g}_{(\alpha\beta|}[M_{g|\gamma)d}] + \sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)gd} - \sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} &- 8\gamma^{2}\sigma_{e(\alpha|\epsilon}\sigma_{f(\beta|\tau}\sigma_{d|\gamma)\phi}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau}T^{(0)ef\phi}\end{aligned} \tag{34}$$

Looking at equation (34) we see that we will encounter an intractable term in the H sector unless we can absorb it into the $T^{(2)}{}_{\beta\gamma}{}^{\lambda}$ term, that is the fourth term on the RHS of (34). Let us call it the T^{3} term. There is no known sigma matrix identity that will allow this term to be written in the form necessary to solve for $H_{g\gamma d}$. We see that the T^{3} term has the same sigma matrix structure as the $T^{(2)}{}_{\beta\gamma}{}^{\lambda}$ term. We consider the option of equating the $T^{(2)}{}_{\beta\gamma}{}^{\lambda}$ term with the T^{3} term. This is possible only when X is non zero, and it will constitute one of the many necessary components of the solution.

4 Results Necessary For Solution

In order to see relationships and recognize cancelations we retain on our simplified index notation, for example $T^{(2)}{}_{\beta\gamma}{}^{\lambda}$, and keep in mind that quantities can be written in several ways which make them recognizable. We employ the following results which are crucial

to obtaining a solution. We will require the zeroth order spinor derivative of $T^{(0)}{}_{ef}{}^{\delta}$, available from (30),

$$\nabla_{\gamma} T^{(0)}{}_{ef}{}^{\delta} = [-\frac{1}{4}\sigma^{mn}{}_{\gamma}{}^{\delta} R_{efmn} \ + \ T^{(0)}{}_{ef}{}^{\lambda} T^{(0)}{}_{\gamma\lambda}{}^{\delta}] \tag{35}$$

We refer to its χ part and its curvature part using a suitable subscript as in (36).

We have the following results, the derivations of which are lengthy and are included in appendices.

$$\begin{aligned} T^{(0)}{}_{(\alpha\beta|}{}^{\lambda} \frac{i\gamma}{6} \sigma^{pqref|}{}_{\gamma)\lambda} A^{(1)}{}_{pqr} H^{(0)}{}_{def} - \frac{i\gamma}{6} \sigma^{pqref}{}_{(\alpha\beta|} [H^{(0)}{}_{def} \nabla_{|\gamma)} A^{(1)}{}_{pqr}] \, |_{(\chi)} & \\ = \ + \ i\gamma \sigma^{g}_{(\alpha\beta|} H^{(0)}{}_{def} \nabla_{|\gamma)} A^{(1)}{}_{gef} \, |_{(\chi)} & \end{aligned} \tag{36}$$

We also require, (See appendix IV.)

$$\begin{aligned} & + \ \frac{i\gamma}{6} \sigma^{pqref}{}_{(\alpha\beta|} H^{(0)}{}_{def} \nabla_{|\gamma)} A^{(1)}{}_{pqr}|_{(R)} = \ \frac{\gamma^2}{12} \sigma^{pqref}{}_{(\alpha\beta|} H^{(0)}{}_{def} \sigma_{pqr\epsilon\tau} T^{(0)}{}_{kp}{}^{\epsilon} \sigma^{mn}{}_{|\gamma)}{}^{\tau} R^{(0)kp}{}_{mn} \\ & = \ \frac{\gamma^2}{2} \sigma^{g}{}_{(\alpha\beta|} \sigma_{gef\epsilon\tau} T^{(0)}{}_{kp}{}^{\epsilon} \sigma^{mn}{}_{|\gamma)}{}^{\tau} R^{(0)kp}{}_{mn} H^{(0)}{}_{d}{}^{ef} + \ 16\gamma^2 \sigma^{g}{}_{(\alpha\beta|} \sigma_{e|\gamma)\epsilon} T^{(0)}{}_{kp}{}^{\epsilon} R^{(0)}{}_{fg}{}^{kp} H^{(0)}{}_{d}{}^{ef} \end{aligned} \tag{37}$$

And for later transparency we have the very important and convenient observation that, using the identity $\sigma^{g}{}_{(\alpha\beta|}\sigma_{g|\gamma)\lambda} = 0$, this can be written as

$$\begin{aligned} & \frac{i\gamma}{6} \sigma^{pqref}{}_{(\alpha\beta|} H^{(0)}{}_{def} \nabla_{|\gamma)} A^{(1)}{}_{pqr}|_{(R)} \\ & = -i\gamma \sigma^{g}{}_{(\alpha\beta|} [\nabla_{|\gamma)} A^{(1)}{}_{gef}]|_{(R)} H^{(0)}{}_{d}{}^{ef} + 4i\gamma \sigma^{g}{}_{(\alpha\beta|} R^{(1)}{}_{|\gamma)gef} H^{(0)}{}_{d}{}^{ef} \end{aligned} \tag{38}$$

Combining the results (36) and (38) we get the very simple reduction,

$$+\frac{i\gamma}{6}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}\sigma^{pqref}{}_{|\gamma)\lambda}A^{(1)}{}_{pqr}H^{(0)}{}_{def}-\frac{i\gamma}{6}\sigma^{pqref}{}_{(\alpha\beta|}H^{(0)}{}_{def}\nabla_{|\gamma)}A^{(1)}{}_{pqr}$$
$$=+i\gamma\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{gef}]H^{(0)}{}_{d}{}^{ef}-4i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gef}H^{(0)}{}_{d}{}^{ef} \tag{39}$$

This will be a key part of the solution. Along with the sigma matrix identities given in [3], we also require the important result that

$$\sigma^{pqref}{}_{(\alpha\beta|}\sigma_{e|\gamma)\phi} = \; = \; - \; \sigma^{pqref}{}_{\phi(\gamma|}\sigma_{e|\alpha\beta)} \tag{40}$$

The latter allows us to write

$$\frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}R^{(0)}{}_{|\gamma)def} = \; +\frac{\gamma}{6}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}\phi \tag{41}$$

Hence we transform it to a solvable term. It is the above system of derived identities and observations, coupled with an Anzatz for the X tensor that will facilitate our solution up to a curvature. Solving the final curvature will involve other technical difficulties.

5 The H Sector Solution

To second order in perturbation theory we have, using our exponent notation, for $H^{(2)}{}_{\beta\gamma d}$

$$\frac{1}{6}\nabla_{(\alpha|}H_{|\beta\gamma\delta)}{}^{Order(2)} \; - \; \frac{1}{4}T^{(0)}{}_{(\alpha\beta|}{}^{g}H^{(2)}{}_{g|\gamma\delta)} \; - \; \frac{1}{4}T^{(1)}{}_{(\alpha\beta|}{}^{g}H^{(1)}{}_{g|\gamma\delta)} \; - \; \frac{1}{4}T^{(2)}{}_{(\alpha\beta|}{}^{g}H^{(0)}{}_{g|\gamma\delta)}$$
$$- \; \frac{1}{4}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}H^{(2)}{}_{\lambda|\gamma\delta)} \; - \; \frac{1}{4}T^{(1)}{}_{(\alpha\beta|}{}^{\lambda}H^{(1)}{}_{\lambda|\gamma\delta)} \; - \; \frac{1}{4}T^{(2)}{}_{(\alpha\beta|}{}^{\lambda}H^{(0)}{}_{\lambda|\gamma\delta)} \; = \; -\frac{\gamma}{2}R^{(1)}{}_{(\alpha\beta|ef}R^{(0)}{}_{|\gamma\delta)}{}^{ef} \tag{42}$$

For $H^{(2)}{}_{\beta gd}$ we have

$$
\begin{aligned}
&\frac{1}{2}\nabla_{(\alpha|}H^{Order\gamma^2}_{|\beta\gamma)d} \;-\; \nabla_d H^{Order\gamma^2}_{\alpha\beta\gamma} \;-\; \frac{1}{2}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}H^{(2)}{}_{\lambda|\gamma)d} \;-\; \frac{1}{2}T^{(1)}{}_{(\alpha\beta|}{}^{\lambda}H^{(1)}{}_{\lambda|\gamma)d} \\
&- \frac{1}{2}T^{(2)}{}_{(\alpha\beta|}{}^{\lambda}H^{(0)}{}_{\lambda|\gamma)d} \;-\; \frac{1}{2}T^{(0)}{}_{(\alpha\beta|}{}^{g}H^{(2)}{}_{g|\gamma)d} \;-\; \frac{1}{2}T^{(1)}{}_{(\alpha\beta|}{}^{g}H^{(1)}{}_{g|\gamma)d} \;-\; \frac{1}{2}T^{(2)}{}_{(\alpha\beta|}{}^{g}H^{(0)}{}_{g|\gamma)d} \\
&+ \frac{1}{2}T^{(2)}{}_{d(\alpha}{}^{g}H^{(0)}{}_{g|\beta\gamma)} \;+\; \frac{1}{2}T^{(1)}{}_{d(\alpha}{}^{g}H^{(1)}{}_{g|\beta\gamma)} + \frac{1}{2}T^{(0)}{}_{d(\alpha}{}^{g}H^{(2)}{}_{g|\beta\gamma)} \\
&+ \gamma'[R^{(0)}{}_{(\alpha\beta|ef}R^{(1)}{}_{|\gamma)d}{}^{ef} \;+\; R^{(1)}{}_{(\alpha\beta|ef}R^{(0)}{}_{|\gamma)d}{}^{ef}] \;=0
\end{aligned}
\tag{43}
$$

We need to solve (42) and (43). We do so using the constraints of reference [4].

It is a straightforward to solve for $H^{(2)}{}_{g\gamma\delta}$. To do so we use the constraints in ref. [4] and substitute them into equation (42). We extract a sigma matrix coefficient from each term as below. After some care with the algebra we obtain.

$$\sigma_{(\alpha\beta|}{}^{g}H^{(2)}{}_{g|\gamma\delta)} \;=\; \sigma^{g}{}_{(\alpha\beta|}[-4\gamma H^{(0)}{}_{gef}R^{(1)}{}_{|\gamma\delta)}{}^{ef} \;-\; \frac{1}{2}T^{(2)}{}_{g|\gamma\delta)}] \tag{44}$$

Which solves with the correct symmetries to simply

$$H^{(2)}{}_{g\gamma\delta} \;=\; [-4\gamma H^{(0)}{}_{gef}R^{(1)}{}_{\gamma\delta}{}^{ef} \;-\; \frac{1}{2}T^{(2)}{}_{g\gamma\delta}] \tag{45}$$

Which we can also write as

$$
\begin{aligned}
H^{(2)}{}_{d\alpha\beta} \;=\; &\sigma_{\alpha\beta}{}^{g}[8i\gamma H^{(0)}{}_{def}L^{(1)}{}_{g}{}^{ef} \;-\; i\gamma H^{(0)}{}_{def}A^{(1)}{}_{g}{}^{ef}] \\
&+ \sigma^{pqref}{}_{\alpha\beta}[\frac{-i\gamma}{6}H^{(0)}{}_{def}A^{(1)}{}_{pqr} \;-\; \frac{1}{2}X_{pqrefd}]
\end{aligned}
\tag{46}
$$

For the next Bianchi Identity we will require considerably more ingenuity. The result

will also have to be extracted using a symmetrization operator, equation (135). Applying the constraints of ref. [4] reduces (43) to

$$\frac{1}{2}\nabla_{(\alpha|}H_{|\beta\gamma)d}{}^{Order\gamma^2} \;-\; \frac{1}{2}T^{(2)}{}_{(\alpha\beta|}{}^{\lambda}H^{(0)}{}_{\lambda|\gamma)d} \;-\; \frac{1}{2}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}H^{(2)}{}_{\lambda|\gamma)d} \;-\; \frac{1}{2}T^{(0)}{}_{(\alpha\beta|}{}^{g}H^{(2)}{}_{g|\gamma)d}$$
$$+\; \frac{1}{2}T^{(2)}{}_{d(\alpha}{}^{g}H^{(0)}{}_{g|\beta\gamma)} \;+\; \gamma'[R^{(0)}{}_{(\alpha\beta|ef}R^{(1)}{}_{|\gamma)d}{}^{ef} \;+\; R^{(1)}{}_{(\alpha\beta|ef}R^{(0)}{}_{|\gamma)d}{}^{ef}] \;=0 \tag{47}$$

We now substitute the constraints known from ref. [4] into (47) as before. After some long calculations and we obtain

$$\begin{aligned}
\frac{+i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} \;=\; & \frac{1}{2}\nabla_{(\alpha|}H_{|\beta\gamma)d}{}^{Order\gamma^2} \;-\; \frac{i}{4}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\alpha\beta)}{}^{\lambda} \\
& +\; \sigma^{g}{}_{(\alpha\beta|}[-2\gamma\sigma_{g}{}^{\lambda\phi}\chi_{\phi}H^{(0)}{}_{def}R^{(1)}{}_{\lambda|\gamma)ef} \\
& -\; \frac{1}{4}\sigma_{g}{}^{\lambda\phi}\chi_{\phi}T^{(2)}{}_{d\lambda|\gamma)}] \;-4\gamma\chi_{(\alpha|}H^{(0)}{}_{def}R^{(1)}{}_{|\beta\gamma)}{}^{\;ef} \\
& -\frac{1}{2}\chi_{(\alpha}T^{(2)}{}_{d|\beta\gamma)} + \frac{i}{4}\sigma^{g}_{(\alpha\beta|}T^{(2)}{}_{d|\gamma)g} - 2i\gamma\sigma^{g}{}_{(\alpha\beta|}H^{(0)}{}_{gef}R^{(1)}{}_{|\gamma)d}{}^{\;ef} \\
& +\; \sigma^{g}{}_{(\alpha\beta|}[-i2\gamma\Pi^{(1)}{}_{g}{}^{ef}R^{(0)}{}_{|\gamma)d}{}^{ef}] + \frac{i\gamma}{24}\sigma^{pqr}{}_{ef(\alpha\beta|}R^{(0)}{}_{|\gamma)d}{}^{ef}A^{(1)}{}_{pqr}
\end{aligned} \tag{48}$$

Equation (48) contains a proliferation of non solvable non linear terms. Hence we follow our first route. We consider the spinor derivative of $H_{\beta\gamma d}{}^{Order\gamma^2}$. Since we do not yet know what the X tensor is we will first employ a torsion and a curvature to eliminate χ and X tensor terms. We retain our compact superscript notation for terms which will later cancel and so we will not write them out in full unless required. We must also remember to include the second order derivative contributions which come from the first

order result. In [4] it was found to first order,

$$H_{\beta\gamma d} \;=\; \frac{i}{2}\sigma_{d\beta\gamma} + i4\gamma\sigma^{g}{}_{\beta\gamma}H^{(0)}{}_{gef}H^{(0)}{}_{def} \tag{49}$$

Taking the spinor derivative of this first order term will generate second order contributions. Hence we include this contribution. However we do not list the terms explicitly. The contribution due to this term is simply a lengthy expression with the correct sigma matrix structure needed to extract the solution. The explicit result will be written in full in a later paper. The method of solution for this Bianchi identity involves extracting a sigma matrix similar to the coefficient of the term which we seek. Namely we seek to solve for $H^{(2)}{}_{g\gamma d}$ in an expression of the form

$$\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} \;=\; \sigma^{g}{}_{(\alpha\beta|}M^{(2)}{}_{g|\gamma)d} \tag{50}$$

Finally we extract the correct expression for $H^{(2)}{}_{g\gamma d}$ with an appropriate symmetrization operator, (section (12)). We have from first order, which we leave as is,

$$\frac{1}{2}\nabla_{(\alpha|}H^{(1)}{}_{|\beta\gamma)d}{}^{(Order\gamma^2)} \;=\; \sigma^{g}_{(\alpha\beta|}(2i\gamma\nabla_{|\gamma)}[H^{(0)}{}_{gef}H^{(0)}{}_{d}{}^{ef}])^{Order(1)} \tag{51}$$

We then have, taking the derivative of (45)

$$\frac{1}{2}\nabla_{(\alpha|}H^{(2)}{}_{|\beta\gamma)d}{}^{(Order\gamma^2)} \;=\; [-2\gamma\nabla_{(\alpha|}(H^{(0)}{}_{def}R^{(1)}{}_{|\beta\gamma)}{}^{ef}) \;-\; \frac{1}{4}\nabla_{(\alpha|}T^{(2)}{}_{d|\beta\gamma)}] \tag{52}$$

$$= (\nabla_{(\alpha|}[-2\gamma H^{(0)}{}_{def}])R^{(1)}{}_{|\beta\gamma)}{}^{ef} \;-\; 2\gamma H^{(0)}{}_{def}[\nabla_{(\alpha|}R^{(1)}{}_{|\beta\gamma)}{}^{ef}] \;-\; \frac{1}{4}\nabla_{(\alpha|}T^{(2)}{}_{d|\beta\gamma)} \tag{53}$$

$$= \sigma^g_{(\alpha\beta|}[4i\gamma(\nabla_{(\alpha|}H^{(0)}{}_{def})\Pi^{(1)}{}_f{}^{ef} - \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\nabla_{|\gamma)}H^{(0)}{}_{def}]$$

$$- \frac{1}{4}\nabla_{(\alpha|}T^{(2)}{}_{d|\beta\gamma)} - 2\gamma H^{(0)}{}_{def}\nabla_{(\alpha|}R^{(1)}{}_{|\beta\gamma)}{}^{ef} \quad (54)$$

We need to evaluate $\nabla_{(\alpha|}T^{(2)}{}_{d|\beta\gamma)}$ and $\nabla_{(\alpha|}R^{(1)}{}_{|\beta\gamma)ef}$. We will later calculate the derivative directly but firstly we use an indirect approach. To do this we use a first order curvature and the dimension one half torsion at second order. This will eliminate the *chi* terms as well as the X tensor terms. However it also will isolate the torsion $T^{(2)}{}_{\alpha\beta}{}^{\lambda}$, and allow us to identify a candidate for this torsion. This candidate in turn will be shown to satisfy the dimension one half torsion, (20).

For the curvature we solve the Bianchi Identity

$$\nabla_{(\alpha|}R^{(1)}{}_{|\beta\gamma)ef} = T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}R^{(1)}{}_{|\gamma)\lambda ef} + T^{(1)}{}_{(\alpha\beta|}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda ef} - T^{(0)}{}_{(\alpha\beta|}{}^{g}R^{(1)}{}_{|\gamma)gef}$$

$$- T^{(1)}{}_{(\alpha\beta|}{}^{g}R^{(0)}{}_{|\gamma)gef} \quad (55)$$

Using the first order constraints of Ref [4] we obtain

$$-2\gamma H^{(0)}{}_d{}^{ef}\nabla_{(\alpha|}R^{(1)}{}_{|\beta\gamma)ef} = \sigma^g{}_{(\alpha\beta|}[2\gamma\sigma_g{}^{\lambda\phi}\chi_\phi R^{(1)}{}_{|\gamma)\lambda ef}H^{(0)}{}_d{}^{ef}$$

$$+ 2i\gamma R^{(1)}{}_{|\gamma)gef}H^{(0)}{}_d{}^{ef}] + 4\gamma\chi_{(\alpha|}R^{(1)}{}_{|\beta\gamma)ef}H^{(0)}{}_d{}^{ef} \quad (56)$$

Similarly for the Torsion we have at second order, using equation (20),

$$\frac{-1}{4}\nabla_{(\alpha|}T^{(2)}{}^{d}{}_{|\beta\gamma)} = \frac{1}{2}\chi_{(\alpha|}T^{(2)}{}_{|\beta\gamma)}{}^{d} - \frac{i}{4}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} + \sigma^g{}_{(\alpha\beta|}[\frac{1}{4}\sigma_g{}^{\lambda\phi}T^{(2)}{}_{|\gamma)\lambda}{}^{d}$$

$$+ \frac{i}{4}T^{(2)}{}_{|\gamma)g}{}^{d}] = 0 \quad (57)$$

Substituting these results into the derivative $\nabla_{(\alpha|}H_{|\beta\gamma)d}{}^{Order\gamma^2}$ gives the complete expression which in turn will cancel the remaining non linear terms in (48) exactly. We have

$$\begin{aligned}\frac{1}{2}\nabla_{(\alpha|}H_{|\beta\gamma)d}{}^{Order2} &= \sigma^g_{(\alpha\beta|}[2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_g{}^{ef})^{Order2} \;+\; 4i\gamma\nabla_{|\gamma)}H^{(0)}{}_{def}\Pi^{(1)}{}_g{}^{ef}] \\ &- \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}[\nabla_{|\gamma)}(H^{(0)}{}_{def})]A^{(1)}{}_{pqr} \;+\; \frac{1}{2}\chi_{(\alpha|}T^{(2)}{}_{|\beta\gamma)d} \\ &- \frac{i}{4}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} \;+\; \sigma^g_{(\alpha\beta|}[(\frac{1}{4})\sigma_g{}^{\lambda\phi}\chi_\phi T^{(2)}{}_{|\gamma)\lambda d} \;+\; \frac{i}{4}T^{(2)}{}_{|\gamma)gd}] \\ &+ \sigma^g{}_{(\alpha\beta|}[2\gamma\sigma_g{}^{\lambda\phi}\chi_\phi R^{(1)}{}_{|\gamma)\lambda ef}H^{(0)}{}_d{}^{ef} \;+\; 2i\gamma H^{(0)}{}_{def}R^{(1)}{}_{|\gamma)g}{}^{ef}] \\ &+ 4\gamma H^{(0)}{}_{def}\chi_{(\alpha|}R^{(1)}{}_{|\beta\gamma)}{}^{ef}\end{aligned} \tag{58}$$

After substitution if the derivative term (58) into (48) we find many cancelations and arrive at a considerably reduced χ free

$$\begin{aligned}\frac{i}{2}\sigma^g{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} = \quad & \sigma^g_{(\alpha\beta|}[+2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_g{}^{ef}) \;-\; 2i\gamma R^{(1)}{}_{|\gamma)[d|}{}^{ef}H^{(0)}{}_{|g]ef} \\ &- 2i\gamma\Pi^{(1)}{}_g{}^{ef}[R^{(0)}{}_{|\gamma)def} \;-\; 2\nabla_{|\gamma)}H^{(0)}{}_{def}]] \\ &+ \frac{i\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[R^{(0)}{}_{|\gamma)def} \;-\; 2\nabla_{|\gamma)}H^{(0)}{}_{def}] \\ &- \frac{i}{2}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} \;+\; \sigma^g{}_{(\alpha\beta|}\frac{i}{2}T^{(2)}{}_{d|\gamma)g}\end{aligned} \tag{59}$$

We write the expression this way for some transparency. Now we consider the sigma five part. Although we do not do it now, we note that the term with $R^{(0)}{}_{\gamma d}{}^{ef}$ allows it to be written as a solvable term because the identity,

$$\sigma^{pqref}{}_{(\alpha\beta|}\sigma_{e|\gamma)\phi} = \;=\; -\;\sigma^{pqref}{}_{\phi(\gamma|}\sigma_{e|\alpha\beta)} \tag{60}$$

Hence we have using the constraints in [4],

$$+\;\frac{i\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[R^{(0)}{}_{|\gamma)def} \;-\; 2\nabla_{|\gamma)}H^{(0)}{}_{def}] \;=\; +\;\frac{\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}\sigma_{d|\gamma)\phi}T_{ef}{}^{\phi} \tag{61}$$

After a long calculation (see appendix V) we can show that

$$
\begin{aligned}
& + \ \frac{\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}\sigma_{d|\gamma)\phi}T_{ef}{}^{\phi} \\
= & - \frac{\gamma}{4}\sigma^{g}{}_{(\alpha\beta}A^{(1)}{}_{g}{}^{ef}\sigma_{d|\gamma)\phi}T_{ef}{}^{\phi} \ + \ 4i\gamma^{2}\sigma_{e(\alpha|\epsilon}\sigma_{f|\beta)\tau}\sigma_{d|\gamma)\phi}T_{kp}{}^{\epsilon}T^{kp\tau}T^{ef\phi}
\end{aligned}
\qquad (62)
$$

It was the second term in RHS of the above expression, (62), that caused the problem of closure. It is not in a solvable form, and it cannot be written as such. We could avoid the problem by absorbing the second term of the above equation into the torsion $T^{(2)\lambda}{}_{\beta\gamma}$. The terms in (62) all have a similar $\sigma_{d\gamma\phi}$ coefficient and also similar to that of $T^{(2)\lambda}{}_{\beta\gamma}$. It remains to be considered as to what combination should be absorbed into $T^{(2)\lambda}{}_{\beta\gamma}$. We could have

$$
\frac{i}{2}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} \ = \ + \ 4i\gamma^{2}\sigma_{e(\alpha|\epsilon}\sigma_{f|\beta)\phi}\sigma_{d|\gamma)\lambda}T_{kp}{}^{\epsilon}T^{kp\phi}T^{ef\lambda} \qquad (63)
$$

However we have evidence from the dimension one half torsion, (20), that we should in fact choose the whole quantity with that particular sigma matrix structure as follows

$$
i\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} = + \ \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[R^{(0)}{}_{|\gamma)def} \ - \ 2\nabla_{|\gamma)}H^{(0)}{}_{def}] \qquad (64)
$$

Or

$$
T^{(2)}{}_{\alpha\beta}{}^{\lambda} \ = \ - \ \frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}T_{ef}{}^{\phi} \qquad (65)
$$

This scenario will give for $H^{(2)}{}_{g\gamma d}$,

$$
\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} = \quad \sigma^{g}{}_{(\alpha\beta|}[+ 2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef}) \ - \ 2i\gamma R^{(1)}{}_{|\gamma)[d|}{}^{ef}H^{(0)}{}_{|g]ef}
$$

$$- 2i\gamma[\Pi^{(1)}{}_g{}^{ef}][R^{(0)}{}_{|\gamma)def} \; - \; 2\nabla_{|\gamma)}H^{(0)}{}_{def}]]$$

$$+ \; \sigma^g{}_{(\alpha\beta|}\frac{i}{2}T^{(2)}{}_{d|\gamma)g} \qquad (66)$$

This is written in terms of the as yet unknown torsion $T^{(2)}{}_{d\gamma g}$.

6 Closure Using the X Tensor

We now propose an Anzatz for the X tensor and show that in conjunction with the results (64), and (39), that we can indeed close the H sector by a different route. We later show that these results close the dimension one half torsion.

We have, using the dimension one half torsion (20),

$$\begin{aligned} & - \; \frac{i}{2}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} \; + \; \sigma^g{}_{(\alpha\beta|}\frac{i}{2}T^{(2)}{}_{d|\gamma)g} \\ & = \; \frac{1}{2}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}T^{(2)}{}_{|\gamma)\lambda}{}^{d} \; - \; \frac{1}{2}\nabla_{(\alpha|}T^{(2)}{}_{|\beta\gamma)}{}^{d} \end{aligned} \qquad (67)$$

The last two terms in equation (59) also appear as in this combination.

We now say let

$$T^{(2)}{}_{\alpha\beta}{}^{d} = \sigma^{pqref}{}_{\alpha\beta}[X_{pqrefd} + Y_{pqrefd}] \qquad (68)$$

where $X_{pqrefd} - \frac{i\gamma}{6}\sigma^{pqref}{}_{\alpha\beta}$. We find that $Y_{pqrefd} = 0$ is sufficient to close the H sector and torsion sector identities. Hence we choose the Ansatz

$$T^{(2)}{}_{\alpha\beta}{}^{d} \; = - \; \frac{i\gamma}{6}\sigma^{pqref}{}_{\alpha\beta}H^{(0)d}{}_{ef}A^{(1)}{}_{pqr} \qquad (69)$$

Therefore

$$- \frac{i}{2}\sigma_{d(\alpha|\lambda}T^{(2)\ \lambda}_{|\beta\gamma)} + \sigma^{g}{}_{(\alpha\beta|}\frac{i}{2}T^{(2)}{}_{d|\gamma)g}$$

$$= \frac{1}{2}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}[- \frac{i\gamma}{6}\sigma^{pqref}{}_{|\gamma)\lambda}H^{(0)}{}_{def}A^{(1)}{}_{pqr}] - \frac{1}{2}\nabla_{(\alpha|}[- \frac{i\gamma}{6}\sigma^{pqref}{}_{|\beta\gamma)}H^{(0)}{}_{def}A^{(1)}{}_{pqr}] \quad (70)$$

$$= \frac{1}{2}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}[- \frac{i\gamma}{6}\sigma^{pqref}{}_{|\gamma)\lambda}H^{(0)}{}_{def}A^{(1)}{}_{pqr}] - \frac{1}{2}[- \frac{i\gamma}{6}\sigma^{pqref}{}_{|\beta\gamma)}H^{(0)}{}_{def}[\nabla_{(\alpha|}A^{(1)}{}_{pqr}]$$
$$- \frac{1}{2}[- \frac{i\gamma}{6}\sigma^{pqref}{}_{|\beta\gamma)}[\nabla_{(\alpha|}H^{(0)}{}_{def}]A^{(1)}{}_{pqr} \quad (71)$$

We now use equation (39) of our recipe to get

$$- \frac{i}{2}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} + \sigma^{g}{}_{(\alpha\beta|}\frac{i}{2}T^{(2)}{}_{d|\gamma)g} = -\frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{gef}]H^{(0)}{}_{d}{}^{ef}$$
$$+2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gef}H^{(0)}{}_{d}{}^{ef} + \frac{i\gamma}{12}[\sigma^{pqref}{}_{(\alpha\beta|}[\nabla_{|\gamma)}H^{(0)}{}_{def}]A^{(1)}{}_{pqr} \quad (72)$$

Incorporating the result (72) into (59) gives

$$\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} = \sigma^{g}_{(\alpha\beta|}[+2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef}) - 2i\gamma R^{(1)}{}_{|\gamma)[d|}{}^{ef}H^{(0)}{}_{|g]ef}$$
$$- 2i\gamma[\Pi^{(1)}{}_{g}{}^{ef}][R^{(0)}{}_{|\gamma)def} - 2\nabla_{|\gamma)}H^{(0)}{}_{def}]]$$
$$+ \frac{i\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[R^{(0)}{}_{|\gamma)def}]$$
$$- \frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}\nabla_{|\gamma)}A^{(1)}{}_{gef}H^{(0)}{}_{d}{}^{ef} + 2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gef}H^{(0)}{}_{d}{}^{ef}$$
$$(73)$$

Furthermore using equation (41) we have

$$\frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}R^{(0)}{}_{\gamma)def} = \ +\frac{\gamma}{6}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}\phi \tag{74}$$

Hence we finally obtain a fully solvable form as follows

$$\begin{aligned}\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} = \sigma^{g}_{(\alpha\beta|}[&+2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef}) - 2i\gamma R^{(1)}{}_{|\gamma)d|}{}^{ef}H^{(0)}{}_{gef} \\ &-2i\gamma[\Pi^{(1)}{}_{g}{}^{ef}][R^{(0)}{}_{|\gamma)def} - 2\nabla_{|\gamma)}H^{(0)}{}_{def}]] \\ &+\frac{\gamma}{12}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}\phi - \frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}\nabla_{|\gamma)}A^{(1)}{}_{gef}H^{(0)}{}_{d}{}^{ef}\end{aligned} \tag{75}$$

Hence we have solved the first part old problem. Also the residual terms in (75), should be those obtained in the solution to the dimension one half torsion. In fact in the next section we find that that is just so, hence adding good support to our Ansatz for $T^{(2)}{}_{\beta\gamma}{}^{g}$ and result for the torsion $T^{(2)}{}_{\beta\gamma}{}^{\lambda}$. We show that this is true in this scenario, and we also show that it is true in the case where we take the direct derivative $H^{(2)}{}_{\alpha\beta d}$ without using the curvature or torsion to eliminate non-linear terms.

7 Dimension One Half Torsion

We now look at the dimension one half torsion, (equation (20)). At second order since all relevant first order quantities are zero this becomes

$$T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}T^{(2)}{}_{|\gamma)\lambda}{}^{d} \ + \ T^{(2)}{}_{(\alpha\beta|}{}^{\lambda}T^{(0)}{}_{|\gamma)\lambda}{}^{d} \ - \ T^{(0)}{}_{(\alpha\beta|}{}^{g}T^{(2)}{}_{|\gamma)g}{}^{d} \ - \ \nabla_{(\alpha|}T^{(2)}{}_{|\beta\gamma)}{}^{d} \ = \ 0 \tag{76}$$

We have the candidate for the X tensor, equation (69). We also had the candidate for the complete term , $i\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda}$, (64). Hence substitution of these results into the torsion (76) gives

$$
\begin{aligned}
T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}\sigma^{pqref}{}_{|\gamma)\lambda}A^{(1)}{}_{pqr}H^{(0)}{}_{def}[-\frac{i\gamma}{6}] + \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}R^{(0)}{}_{\gamma)def} \\
- \frac{i\gamma}{6}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}\nabla_{|\gamma)}H^{(0)}{}_{def} + \frac{i\gamma}{6}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}\nabla_{|\gamma)}H^{(0)}{}_{def} \\
+ \frac{i\gamma}{6}\sigma^{pqref}{}_{(\alpha\beta|}H^{(0)}{}_{def}\nabla_{|\gamma)}A^{(1)}{}_{pqr}\mid_{\chi} + \frac{i\gamma}{6}\sigma^{pqref}{}_{(\alpha\beta|}H^{(0)}{}_{def}\nabla_{|\gamma)}A^{(1)}{}_{pqr}\mid_{R} \\
-i\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)gd} = 0
\end{aligned}
\tag{77}
$$

Again using (39) and (41) gives many cancelations of terms which otherwise would be intractable. Hence we obtain the very short result,

$$
\begin{aligned}
+\ i\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)gd} \ = -i\gamma\sigma^{g}{}_{(\alpha\beta|}\nabla_{|\gamma)}A^{(1)}{}_{gef}H^{(0)}{}_{d}{}^{ef} \\
4i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gef}H^{(0)}{}_{def} \ + \ \frac{\gamma}{6}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}{}^{\phi}
\end{aligned}
\tag{78}
$$

This is in agreement with comparing (59) to (75).

8 Direct Derivative Check

We now look at an extremely interesting check on our work. It will be based on a previously unnoticed observation. We began by calculating $H^{(2)}{}_{\beta\gamma d}$. We found this quite easily. It contained the X tensor. We needed the spinor derivative of $H^{(2)}{}_{\beta\gamma d}$ to find $H^{(2)}{}_{\alpha bg}$. Since we did not know the form or the X tensor we eliminated it as well as all

χ terms by using the dimension one half torsion and also a curvature to replace those terms. Hence we did not calculate this derivative directly. We found a solution for $H^{(2)}{}_{\alpha bg}$ which contained $T^{(2)}{}_{\alpha b}{}^{g}$ as an unknown. In the process we also identified $T^{(2)}{}_{\beta\gamma}{}^{\lambda}$. We then proposed a candidate for the X tensor and showed that it could successfully close the second H sector Bianchi Identities and the dimension one half torsion. This candidate also closed the dimension one half torsion, and produced a result for $T^{(2)}{}_{\alpha b}{}^{g}$. We also found the exact same result for $T^{(2)}{}_{\alpha b}{}^{g}$ by comparing the two results for $H^{(2)}{}_{\alpha bg}$.

In the following we show agreement and consistency with the results of the solution to the dimension one half torsion, and our results in the H sector, while taking the direct derivative of $H^{(2)}{}_{\alpha\beta g}$.

We make the very convenient observation that the following quantities are interchangeable. We find the following useful result which allows us to make a comparison between the direct method and the method used in part (5) that employed a torsion and curvature to eliminate non linear terms. We note conveniently that

$$\nabla_{\alpha} L^{(1)}{}_{abc} = i\gamma\sigma_{[g|\gamma\phi} T^{(0)}{}_{kl}{}^{\phi} R^{kl}{}_{|ef]} \tag{79}$$

Using the result $\sigma^{g}{}_{(\alpha\beta|}\sigma_{g|\gamma)\lambda} = 0$ along with the result for $R^{(1)}{}_{\gamma abc}$ as given in [4], we can write

$$\nabla_{\gamma} L^{(1)}{}_{abc} = R^{(1)}{}_{\gamma abc} \tag{80}$$

This is a crucial observation which will have other roles. We note here that the result

for the same derivative in [2] will fail to work in the following closure, whereas that in [4] will indeed work, and so the coefficient in [4] seems to be correct.

We found the H sector solution, equation (59). We can write it as follows

$$\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} = \sigma^{g}{}_{(\alpha\beta|}[+2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef}) - 2i\gamma R^{(1)}{}_{|\gamma)d}{}^{ef}H^{(0)}{}_{gef}$$
$$+ 2i\gamma R^{(1)}{}_{|\gamma)g}{}^{ef}H^{(0)}{}_{def} - 2i\gamma\Pi^{(1)}{}_{g}{}^{ef}R^{(0)}{}_{|\gamma)def} + 4i\gamma L^{(1)}{}_{g}{}^{ef}\nabla_{|\gamma)}H^{(0)}{}_{def}$$
$$-\frac{i\gamma}{2}A^{(1)}{}_{g}{}^{ef}\nabla_{|\gamma)}H^{(0)}{}_{def} + \sigma^{g}{}_{(\alpha\beta|}\frac{i}{2}T^{(2)}{}_{d|\gamma)g} \tag{81}$$

Also we found

$$+ \frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)gd} = -\frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}\nabla_{|\gamma)}A^{(1)}{}_{gef}H^{(0)}{}_{d}{}^{ef}$$
$$+ 2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gef}H^{(0)}{}_{d}{}^{ef} + \frac{\gamma}{12}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}{}^{\phi} \tag{82}$$

Hence we have

$$\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} = \sigma^{g}_{(\alpha\beta|}[+2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef})$$
$$- 2i\gamma R^{(1)}{}_{|\gamma)d}{}^{ef}H^{(0)}{}_{gef} + 4i\gamma R^{(1)}{}_{|\gamma)g}{}^{ef}H^{(0)}{}_{def}$$
$$-2i\gamma L^{(1)}{}_{g}{}^{ef}R^{(0)}{}_{|\gamma)def} + \frac{i\gamma}{4}A^{(1)}{}_{g}{}^{ef}R^{(0)}{}_{|\gamma)d}{}^{ef}$$
$$+4i\gamma L^{(1)}{}_{g}{}^{ef}\nabla_{|\gamma)}H^{(0)}{}_{def} - \frac{i\gamma}{2}A^{(1)}{}_{g}{}^{ef}\nabla_{|\gamma)}H^{(0)}{}_{def}$$
$$-\frac{i\gamma}{2}\nabla_{|\gamma)}A^{(1)}{}_{gef}H^{(0)}{}_{d}{}^{ef} + \frac{\gamma}{12}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}{}^{\phi}] \tag{83}$$

In terms of the X tensor we obtained

$$
\begin{aligned}
-\ \frac{i}{4}\sigma_{d(\alpha|\lambda}T^{(2)\ \lambda}_{|\beta\gamma)} \ +\ \sigma^{g}{}_{(\alpha\beta|}\frac{i}{4}T^{(2)}{}_{d|\gamma)g} \ &=\ \frac{1}{4}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}[-\ \frac{i\gamma}{6}\sigma^{pqref}{}_{|\gamma)\lambda}H^{(0)}{}_{def}A^{(1)}{}_{pqr}] \\
&\quad -\ \frac{1}{4}\nabla_{(\alpha|}[-\ \frac{i\gamma}{6}\sigma^{pqref}{}_{|\beta\gamma)}H^{(0)}{}_{def}A^{(1)}{}_{pqr}] \qquad (84)
\end{aligned}
$$

Working out (47) term by term by substituting in the first order constraints as before and recalling that $\sigma^{g}{}_{(\alpha\beta|}\sigma_{g|\gamma)\phi} = 0$ eliminates one term, we find after a lengthy calculation that

$$
\begin{aligned}
\frac{+i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} \ &=\ \frac{1}{2}\nabla_{(\alpha|}H_{|\beta\gamma)d}{}^{Order\gamma^2} \ -\ \frac{i}{4}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\alpha\beta)}{}^{\lambda} \ +\ \frac{i}{4}\sigma^{g}_{(\alpha\beta|}T^{(2)}{}_{d|\gamma)g} \\
+\ T^{(0)\ \lambda}_{(\alpha\beta|}\sigma^{pqr}{}_{ef|\gamma)\lambda}[+\ \frac{i\gamma}{12}H^{(0)}{}_{def}A^{(1)}{}_{pqr} \ &+\ \frac{1}{4}X_{pqrefd}] \ +\ \sigma^{g}{}_{(\alpha\beta|}[-2i\gamma H^{(0)}{}_{gef}R^{(1)}{}_{|\gamma)d}{}^{ef} \\
&\quad -2i\gamma\Pi^{(1)}{}_{gef}R^{(0)}{}_{|\gamma)d}{}^{ef}] + \ \frac{i\gamma}{24}\sigma^{pqr}{}_{ef(\alpha\beta|}R^{(0)}{}_{|\gamma)d}{}^{ef}A^{(1)}{}_{pqr}
\end{aligned}
\qquad (85)
$$

We need $\frac{1}{2}\nabla_{(\alpha|}H_{|\beta\gamma)d}{}^{Order\gamma^2}$. We take the derivative of (46) directly. The derivative is

$$
\begin{aligned}
\frac{1}{2}\nabla_{(\alpha|}H_{|\beta\gamma)d}{}^{Order2} \ &=\ \sigma^{g}_{(\alpha\beta|}[2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef})^{Order(2))} \\
+\ 4i\gamma\nabla_{|\gamma)}[H^{(0)}{}_{def}\Pi^{(1)}{}_{g}{}^{ef}] \ &+\frac{1}{2}\sigma^{pqref}{}_{(\alpha\beta|}\nabla_{|\gamma)}[(-\frac{i\gamma}{6}H^{(0)}{}_{def})A^{(1)}{}_{pqr} \\
&\quad +\ \frac{1}{2}X_{pqref}]
\end{aligned}
\qquad (86)
$$

The whole thing becomes

$$
\frac{+i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} \ =\ -\ \frac{i}{4}\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\alpha\beta)}{}^{\lambda} \ +\ \frac{i}{4}\sigma^{g}_{(\alpha\beta|}T^{(2)}{}_{d|\gamma)g}
$$

$$+\ \sigma^{g}_{(\alpha\beta|}[2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef})^{Order(2)}\ +\ 4i\gamma\nabla_{|\gamma)}[H^{(0)}{}_{def}\Pi^{(1)}{}_{g}{}^{ef}]]$$

$$+\frac{1}{2}\sigma^{pqref}{}_{(\alpha\beta|}\nabla_{|\gamma)}[(-\frac{i\gamma}{6}H^{(0)}{}_{def})A^{(1)}{}_{pqr}\ +\ \frac{1}{2}X_{pqref}]$$

$$+\ T^{(0)}_{(\alpha\beta|}{}^{\lambda}\sigma^{pqref}{}_{|\gamma)\lambda}[+\ \frac{i\gamma}{12}H^{(0)}{}_{def}A^{(1)}{}_{pqr}\ +\ \frac{1}{4}X_{pqrefd}]\ +\ \sigma^{g}{}_{(\alpha\beta|}[-2i\gamma H^{(0)}{}_{gef}R^{(1)}{}_{|\gamma)d}{}^{ef}$$

$$-2i\gamma\Pi^{(1)}{}_{gef}R^{(0)}{}_{|\gamma)d}{}^{ef}]+\ \frac{i\gamma}{24}\sigma^{pqr}{}_{ef(\alpha\beta|}R^{(0)}{}_{|\gamma)d}{}^{ef}A^{(1)}{}_{pqr} \tag{87}$$

Substituting in for the X tensor and gives

$$\frac{+i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d}\ =\ \frac{1}{4}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}[-\ \frac{i\gamma}{6}\sigma^{pqref}{}_{|\gamma)\lambda}H^{(0)}{}_{def}A^{(1)}{}_{pqr}]$$

$$-\ \frac{1}{4}\nabla_{(\alpha|}[-\ \frac{i\gamma}{6}\sigma^{pqref}{}_{|\beta\gamma)}H^{(0)}{}_{def}A^{(1)}{}_{pqr}]\ +\ \sigma^{g}_{(\alpha\beta|}[2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef})^{Order(2)}$$

$$+\ 4i\gamma\nabla_{|\gamma)}[H^{(0)}{}_{def}\Pi^{(1)}{}_{g}{}^{ef}]\ +\sigma^{pqref}{}_{(\alpha\beta|}\nabla_{|\gamma)}[(-\frac{i\gamma}{24}H^{(0)}{}_{def})A^{(1)}{}_{pqr}]$$

$$+\ T^{(0)}_{(\alpha\beta|}{}^{\lambda}\sigma^{pqref}{}_{|\gamma)\lambda}[+\ \frac{i\gamma}{24}H^{(0)}{}_{def}A^{(1)}{}_{pqr}]\ +\ \sigma^{g}{}_{(\alpha\beta|}[-2i\gamma H^{(0)}{}_{gef}R^{(1)}{}_{|\gamma)d}{}^{ef}$$

$$-2i\gamma\Pi^{(1)}{}_{gef}R^{(0)}{}_{|\gamma)d}{}^{ef}]+\ \frac{i\gamma}{24}\sigma^{pqr}{}_{ef(\alpha\beta|}R^{(0)}{}_{|\gamma)d}{}^{ef}A^{(1)}{}_{pqr} \tag{88}$$

Terms neatly cancel to get

$$\frac{+i}{2}\sigma^{g}{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d}\ =\ +\ \sigma^{g}_{(\alpha\beta|}[2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef})^{Order(2)}$$

$$+\ 4i\gamma\nabla_{|\gamma)}[H^{(0)}{}_{def}\Pi^{(1)}{}_{g}{}^{ef}]\ -\ 2i\gamma H^{(0)}{}_{gef}R^{(1)}{}_{|\gamma)d}{}^{ef}$$

$$-2i\gamma\Pi^{(1)}{}_{gef}R^{(0)}{}_{|\gamma)d}{}^{ef}]+\ \frac{i\gamma}{24}\sigma^{pqr}{}_{ef(\alpha\beta|}R^{(0)}{}_{|\gamma)d}{}^{ef}A^{(1)}{}_{pqr} \tag{89}$$

Using

$$\sigma^{pqref}{}_{(\alpha\beta|}\sigma_{e|\gamma)\phi} =\ =\ -\ \sigma^{pqref}{}_{(\gamma|\phi}\sigma_{e|\alpha\beta)} \tag{90}$$

Gives the soluble form

$$\begin{aligned}\frac{+i}{2}\sigma^g{}_{(\alpha\beta|}H^{(2)}_{g|\gamma)d} \;&=\; +\;\sigma^g_{(\alpha\beta|}[2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_g{}^{ef})^{Order(2)} \\ &+\;4i\gamma\nabla_{|\gamma)}[H^{(0)}{}_{def}\Pi^{(1)}{}_g{}^{ef}] \;-\; 2i\gamma H^{(0)}{}_{gef}R^{(1)}{}_{|\gamma)d}{}^{ef} \\ &-2i\gamma\Pi^{(1)}{}_{gef}R^{(0)}{}_{|\gamma)d}{}^{ef}] + \;\frac{\gamma}{12}\sigma^g{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|g)\phi}T^{(0)}{}_{de}{}^fA^{(1)}{}_{pqr}\end{aligned} \tag{91}$$

$$\begin{aligned}\Rightarrow \frac{+i}{2}\sigma^g{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} \;&=\; +\;\sigma^g_{(\alpha\beta|}[2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_g{}^{ef})^{Order(2)} \;+\; 4i\gamma\nabla_{|\gamma)}[H^{(0)}{}_{def}]\Pi^{(1)}{}_g{}^{ef} \\ &+\;4i\gamma H^{(0)}{}_{def}\nabla_{|\gamma)}[\Pi^{(1)}{}_g{}^{ef}] \;-\; 2i\gamma H^{(0)}{}_{gef}R^{(1)}{}_{|\gamma)d}{}^{ef} - 2i\gamma\Pi^{(1)}{}_{gef}R^{(0)}{}_{|\gamma)d}{}^{ef}] \\ &+\;\frac{\gamma}{12}\sigma^g{}_{(\alpha\beta|}\sigma^{pqre}{}_{g\gamma)\phi}T^{(0)}{}_{de}{}^{\phi}A^{(1)}{}_{pqr}\end{aligned} \tag{92}$$

The two results (66) and (92) at first sight do not seem to coincide. We need to work out the Π derivative in equation (92) to enable a comparison. For this our crucial observation is given in (80). We have therefore

$$\begin{aligned}4i\gamma\sigma^g{}_{(\alpha\beta|}H^{(0)}{}_{def}\nabla_{|\gamma)}[\Pi^{(1)}{}_g{}^{ef}] \;&=\; 4i\gamma\sigma^g{}_{(\alpha\beta|}H^{(0)}{}_{def}\nabla_{|\gamma)}[L^{(1)}{}_g{}^{ef} \;-\; \frac{1}{8}A^{(1)}{}_g{}^{ef}] \\ &=\; +4i\gamma\sigma^g{}_{(\alpha\beta|}H^{(0)}{}_d{}^{ef}R^{(1)}{}_{|\gamma)gef} \;-\; \frac{i\gamma}{2}\sigma^g{}_{(\alpha\beta|}H^{(0)}{}_d{}^{ef}\nabla_{|\gamma)}A^{(1)}{}_{gef}\end{aligned} \tag{93}$$

$$\begin{aligned}\Rightarrow \frac{+i}{2}\sigma^g{}_{(\alpha\beta|}H^{(2)}{}_{g|\gamma)d} \;&=\; +\;\sigma^g_{(\alpha\beta|}[2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_g{}^{ef})^{Order(2)} \;+\; 4i\gamma(\nabla_{|\gamma)}H^{(0)}{}_{def})\Pi^{(1)}{}_g{}^{ef} \\ &+4i\gamma H^{(0)}{}_d{}^{ef}R^{(1)}{}_{|\gamma)gef} \;-\; \frac{i\gamma}{2}H^{(0)}{}_d{}^{ef}\nabla_{|\gamma)}A^{(1)}{}_{gef}]\end{aligned}$$

$$- \; 2i\gamma H^{(0)}{}_{gef} R^{(1)}{}_{|\gamma)d}{}^{ef} - 2i\gamma \Pi^{(1)}{}_{gef} R^{(0)}{}_{|\gamma)d}{}^{ef}] + \; \frac{\gamma}{12}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi} T^{(0)}{}_{de}{}^{\phi} A^{(1)}{}_{pqr} \tag{94}$$

$$\begin{aligned} = \; \sigma^{g}_{(\alpha\beta|}[+2i\gamma\nabla_{|\gamma)}(H^{(0)}{}_{def}H^{(0)}{}_{g}{}^{ef}) \; - \; 2i\gamma R^{(1)}{}_{|\gamma)[d|}{}^{\;ef} H^{(0)}{}_{|g]ef} \\ - \; 2i\gamma[\Pi^{(1)}{}_{g}{}^{ef}][R^{(0)}{}_{|\gamma)d}{}^{ef} \; - \; 2\nabla_{|\gamma)}H^{(0)}{}_{def}]] \\ + \frac{\gamma}{12}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}\phi \; - \; \frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}\nabla_{|\gamma)}A^{(1)}{}_{gef}H^{(0)}{}_{d}{}^{ef} \end{aligned} \tag{95}$$

Hence we find exact agreement with between (66) and (95), using also (78).

9 Torsion for $T^{(2)}{}_{\alpha d}{}^{\delta}$ at Dimension One

This is based on the constraints listed in [2]. We have the Bianchi identity at dimension one as follows,

$$T_{(\alpha\beta|}{}^{\lambda}T_{|\gamma)\lambda}{}^{\delta} \; - \; T_{(\alpha\beta|}{}^{g}T_{|\gamma)g}{}^{\delta} \; - \; \nabla_{(\alpha|}T_{|\beta\gamma)}{}^{\delta} - \; \frac{1}{4}R_{(\alpha\beta|de}\sigma^{de}{}_{|\gamma)}{}^{\delta} \; = \; 0 \tag{96}$$

At second order it becomes

$$\begin{aligned} T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}T^{(2)}{}_{|\gamma)\lambda}{}^{\delta} \; + \; T^{(2)}{}_{(\alpha\beta|}{}^{\lambda}T^{(0)}{}_{|\gamma)\lambda}{}^{\delta} - i\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)g}{}^{\delta} \; - \; \nabla_{(\alpha|}T^{(2)}{}_{|\beta\gamma)}{}^{\delta} \\ - \; \nabla_{(\alpha|}T^{(0)}{}_{|\beta\gamma)}{}^{\delta(Order\gamma^2)} - \; \frac{1}{4}R^{(2)}{}_{(\alpha\beta|de}\sigma^{de}{}_{|\gamma)}{}^{\delta} \; = \; 0 \end{aligned} \tag{97}$$

We must take care not to neglect second order contributions from the derivative $\nabla_\alpha T^{(0)}{}_{\beta\gamma}{}^{\delta}$.

We have

$$- \ \nabla_{(\alpha|} T^{(0)}{}_{|\beta\gamma)}{}^{\delta(Order\gamma^2)} = [2\delta_{(\alpha|}{}^{\delta}\delta_{|\beta)}{}^{\lambda} + \sigma^{g}{}_{(a\beta|}\sigma_{g}{}^{\delta\lambda}]\nabla_{|\gamma)}\chi_\lambda \tag{98}$$

Using the form of the derivative $\nabla_\alpha \chi_\beta$ as quoted in reference [4] and the result that

$$[2\delta^{\delta}_{(\alpha|}\delta_{|\beta)}{}^{\lambda} + \sigma^{g}{}_{(a\beta|}\sigma_{g}{}^{\delta\lambda}]\sigma_{b|\gamma)\delta} = 0 \tag{99}$$

Gives

$$\begin{aligned} - \nabla_{(\alpha|} T^{(0)Order\gamma^2}{}_{|\beta\gamma)}{}^{\delta} = [2\delta_{(\alpha|}{}^{\delta}\delta_{|\beta)}{}^{\lambda} + \sigma^{g}{}_{(\alpha\beta|}\sigma_{g}{}^{\delta\lambda}]\nabla_{|\gamma)}\chi_\lambda = \\ -\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}\sigma^{mn}{}_{|\gamma)}{}^{\delta}[L^{(2)}{}_{gmn} + \frac{1}{4}A^{(2)}{}_{gmn}] \end{aligned} \tag{100}$$

We now have as before which we re-list for convenience,

$$T^{(2)}{}_{\alpha\beta}{}^{\lambda} \ = \ -\frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}T^{(0)}{}_{ef}{}^{\lambda} \tag{101}$$

Hence the torsion becomes

$$\begin{aligned} T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}[-\frac{i\gamma}{12}\sigma^{pqref}{}_{|\gamma)\lambda}A^{(1)}{}_{pqr}T^{(0)}{}_{ef}{}^{\delta}] \ - \ \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}T^{(0)}{}_{ef}{}^{\delta}T^{(0)}{}_{|\gamma)\lambda}{}^{\delta} - i\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)g}{}^{\delta} \\ + \ \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}(\nabla_{|\gamma)}A^{(1)}{}_{pqr})T^{(0)}{}_{ef}{}^{\delta} + \ \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}(\nabla_{|\gamma)}T^{(0)}{}_{ef}{}^{\delta}) - \ \frac{1}{4}R^{(2)}{}_{(\alpha\beta|de}\sigma^{de}{}_{|\gamma)}{}^{\delta} \\ -\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}\sigma^{mn}{}_{|\gamma)}{}^{\delta}[L^{(2)}{}_{gmn} + \frac{1}{4}A^{(2)}{}_{gmn}] \\ = 0 \end{aligned} \tag{102}$$

Using (39) and (41) again gives gives

$$
\begin{aligned}
-\frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{g}{}^{ef}]T^{(0)}{}_{ef}{}^{\delta}+2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)g}{}^{ef}T^{(0)}{}_{ef}{}^{\delta} \\
-\ \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}T^{(0)}{}_{ef}{}^{\delta}T^{(0)}{}_{|\gamma)\lambda}{}^{\delta}-i\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)g}{}^{\delta} \\
+\ \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[-\frac{1}{4}\sigma^{mn}{}_{|\gamma)}{}^{\delta}R_{efmn}\ +\ T^{(0)}{}_{ef}{}^{\lambda}T^{(0)}{}_{|\gamma)\lambda}{}^{\delta}]-\ \frac{1}{4}R^{(2)}{}_{(\alpha\beta|de}\sigma^{de}{}_{|\gamma)}{}^{\delta} \\
-\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}\sigma^{mn}{}_{|\gamma)}{}^{\delta}[L^{(2)}{}_{gmn}+\frac{1}{4}A^{(2)}{}_{gmn}]=0
\end{aligned}
\qquad (103)
$$

The use of the derivative (35) allows a cancelation of what would otherwise have been an insolvable term. We find

$$
\begin{aligned}
-\frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{g}{}^{ef}]T^{(0)}{}_{ef}{}^{\delta}+2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)g}{}^{ef}T^{(0)}{}_{ef}{}^{\delta} \\
-i\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{|\gamma)g}{}^{\delta}+\ \frac{i\gamma}{12}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[-\frac{1}{4}\sigma^{mn}{}_{|\gamma)}{}^{\delta}R_{efmn}]-\ \frac{1}{4}R^{(2)}{}_{(\alpha\beta|mn}\sigma^{mn}{}_{|\gamma)}{}^{\delta} \\
-\frac{i}{2}\sigma^{g}{}_{(\alpha\beta|}\sigma^{mn}{}_{|\gamma)}{}^{\delta}[L^{(2)}{}_{gmn}+\frac{1}{4}A^{(2)}{}_{gmn}]=0
\end{aligned}
\qquad (104)
$$

From which we read

$$
T^{(2)}{}_{\gamma g}{}^{\delta}=-\frac{\gamma}{2}[\nabla_{\gamma}A^{(1)}{}_{gef}]T^{(0)}{}_{ef}{}^{\delta}+2R^{(1)}{}_{\gamma gef}T^{(0)}{}_{ef}{}^{\delta} \qquad (105)
$$

And

$$
R^{(2)}{}_{\alpha\beta de}=-\ \frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}R_{efde}-2i\sigma^{g}{}_{\alpha\beta}[L^{(2)}{}_{gde}+\frac{1}{4}A^{(2)}{}_{gde}] \qquad (106)
$$

10 Curvature for $R^{(2)}{}_{\lambda gde}$

We need to solve the curvature that will give $R^{(2)}{}_{\lambda gde}$. We have the curvature Bianchi identity which we need to solve at second order as follows

$$T_{(\alpha\beta|}{}^{\lambda} R_{|\gamma)\lambda de} \;-\; T_{(\alpha\beta|}{}^{g} R_{|\gamma)gde} \;-\; \nabla_{(\alpha|} R_{\beta\gamma)de} \;=\; 0 \tag{107}$$

Here we must consider second order contributions from the spinor derivative $\nabla_{(\alpha|} R_{|\beta\gamma)de}$. We write the full curvature to second order for clarity

$$R_{\beta\gamma de} = \; -2i\sigma^{g}{}_{\alpha\beta}\Pi'{}_{gde} + \;\frac{i}{24}\sigma^{pqr}{}_{de\alpha\beta}A^{(1)}{}_{pqr} - \;\frac{i\gamma}{12}\sigma^{pqrab}{}_{\alpha\beta}A^{(1)}{}_{pqr}R_{deab} \tag{108}$$

Where Π' is the modified Π, but is any case is of the solvable form.

$$\Pi' = [L^{(0)}{}_{gde} + L^{(1)}{}_{gde} + L^{(2)}{}_{gde} - \frac{1}{4}A^{(1)}{}_{gde} + \frac{1}{4}A^{(2)}{}_{gde}] \tag{109}$$

With hindsight and in order to eliminate an apparently intractable term we begin by making the following observations. In equation (65) we found $T^{(2)}{}_{\alpha\beta}{}^{\lambda}$, hence we also encountered the quantity

$$\sigma_{d(\alpha|\lambda}T^{(2)}{}_{\alpha\beta}{}^{\lambda} \;=\; -\frac{i\gamma}{12}\sigma_{d(\alpha|\lambda}\sigma^{pqref}{}_{|\beta\gamma)}A^{(1)}{}_{pqr}T^{(0)\lambda}_{ef} \tag{110}$$

Using the torsion, equation (20), we can write

$$\sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} \;=-\;\; \sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{d|\gamma)g}$$

$$+\; iT^{(0)}{}_{(\alpha\beta|}{}^{\lambda}T^{(2)}{}_{|\gamma)\lambda}{}^{d} \;-i\nabla_{(\alpha|}T^{(2)}{}_{|\beta\gamma)}{}^{d} \tag{111}$$

Using the second order torsion results which we found, (65) and (105), then gives

$$\begin{aligned} &+\sigma_{d(\alpha|\lambda}T^{(2)\;\lambda}_{|\beta\gamma)} \;=+\; \sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{d|\gamma)g} \\ +\frac{\gamma}{6}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}[\sigma^{pqref}{}_{|\gamma)\lambda}H^{(0)}{}_{def}A^{(1)}{}_{pqr}] \;-\; &\frac{\gamma}{6}\sigma^{pqref}{}_{(\alpha\beta|}H^{(0)}{}_{def}[\nabla_{|\gamma)}A^{(1)}{}_{pqr}] \\ -\; &\frac{\gamma}{6}\sigma^{pqref}{}_{(\alpha\beta|}[\nabla_{|\gamma)}H^{(0)}{}_{def}]A^{(1)}{}_{pqr} \end{aligned} \tag{112}$$

Now applying our key equation, (39) to (112) gives

$$\begin{aligned} &+\sigma_{d(\alpha|\lambda}T^{(2)\;\lambda}_{|\beta\gamma)} \;=+\; \sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{d|\gamma)g} \\ &+\;\gamma\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{gef}]H^{(0)}{}_{d}{}^{ef} - 4\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gef}H^{(0)}{}_{d}{}^{ef} \end{aligned} \tag{113}$$

We now substitute in our result for $\sigma^{g}{}_{(\alpha\beta|}T^{(2)}{}_{d|\gamma)g}$, (78), into (113) to obtain cancelations and the simple result

$$\begin{aligned} \sigma_{d(\alpha|\lambda}T^{(2)}{}_{|\beta\gamma)}{}^{\lambda} \;&=-\;\frac{i\gamma}{6}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}T^{(0)}{}_{de}{}^{\phi} \\ &=-\frac{i\gamma}{12}\sigma_{d(\alpha|\lambda}\sigma^{pqref}{}_{|\beta\gamma)}A^{(1)}{}_{pqr}T^{(0)\lambda}_{ef} \end{aligned} \tag{114}$$

From which we extract the following result.

$$\sigma^{pqref}{}_{(\alpha\beta|}\sigma_{d|\gamma)|\phi}A^{(1)}{}_{pqr}M_{ef}{}^{\phi} = 2\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqre}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}M_{de}{}^{\phi} \tag{115}$$

From this we deduce a hitherto unknown identity, albeit indirectly, where $M_{de}{}^{\phi}$ is anti-symmetric in 'd' and 'e'. Using our second order torsion and curvature results, (65), (69), and (106), the full curvature at second order becomes

$$\begin{aligned} T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}[-\frac{i\gamma}{12}\sigma^{pqrab}{}_{|\gamma)\lambda}A^{(1)}{}_{pqr}R^{(0)}{}_{abde}] \;+\; [-\frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}T_{ab}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda de} \\ -\; i\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde} \;+\; \frac{i\gamma}{6}\sigma^{pqrab}{}_{(\alpha\beta|}H^{(0)g}{}_{ab}A^{(1)}{}_{pqr}]R^{(0)}{}_{|\gamma)gde} \\ +\; \frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}[\nabla_{|\gamma)|}A^{(1)}{}_{pqr}]R^{(0)}{}_{abde} + \frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\nabla_{|\gamma)}R^{(0)}{}_{abde}] \\ -\; \frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{pqr}] + \; 2i\gamma\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}\Pi'{}_{gde}] = \; 0 \end{aligned} \tag{116}$$

Using (39) again gives two more solvable terms

$$\begin{aligned} +\frac{i\gamma}{12}T^{(0)}{}_{(\alpha\beta|}{}^{\lambda}\sigma^{pqr}{}_{ab|\gamma)\lambda}A^{(1)}{}_{pqr}R^{(0)ab}{}_{de} - \frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}R^{(0)}{}_{abde}\nabla_{|\gamma)}A^{(1)}{}_{pqr} \\ = +\frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{gab}]R^{(0)ab}{}_{de} - 2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)g}{}^{ab}R^{(0)}{}_{abde} \end{aligned} \tag{117}$$

This reduces (116) to

$$\begin{aligned} +\frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}\nabla_{|\gamma)}A^{(1)}{}_{gab}R^{(0)ab}{}_{de} - 2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gab}R^{(0)}{}_{abde} \\ -\frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}T_{ab}{}^{\lambda}R^{(0)}{}_{|\gamma)\lambda de} \end{aligned}$$

$$
\begin{aligned}
- \ i\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde} \ + \ \frac{i\gamma}{6}\sigma^{pqrab}{}_{(\alpha\beta|}H^{(0)g}{}_{ab}A^{(1)}{}_{pqr}R^{(0)}{}_{|\gamma)gde} \\
+ \ \frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\nabla_{|\gamma)}R^{(0)}{}_{abde}] \\
- \ \frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{pqr}] + \ 2i\gamma\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}\Pi'{}_{gde}] = \ 0
\end{aligned}
\tag{118}
$$

We now list the sigma five terms separately.

$$
+\frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[-T^{(0)}{}_{ab}{}^{\lambda}R^{(0)}{}_{\lambda|\gamma)de} \ + \ 2H^{(0)}{}_{ab}{}^{g}R^{(0)}{}_{|\gamma)gde} \ + \ \nabla_{|\gamma)}R^{(0)}{}_{abde}] \tag{119}
$$

$$
= +\frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[-T^{(0)}{}_{ab}{}^{\lambda}R^{(0)}{}_{\lambda|\gamma)de} \ - \ T^{(0)}{}_{ab}{}^{g}R^{(0)}{}_{\gamma gde} \ + \ \nabla_{|\gamma)}R^{(0)}{}_{abde}] \tag{120}
$$

We have the Bianchi Identity

$$
\nabla_{\alpha}R_{abde} \ - \ T_{\alpha[a|}{}^{X}R_{X|b]de} \ - \ T_{ab}{}^{X}R_{X\alpha de} \ + \ \nabla_{[a|}R_{|b]\alpha de} = 0 \tag{121}
$$

The second term on the LHS of (121) is zero at zeroth order. Hence we have as follows,

$$
\begin{aligned}
+ \frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\nabla_{|\gamma)}R^{(0)}{}_{abde}] = \\
+\frac{i\gamma}{12}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[+T^{(0)}{}_{ab}{}^{\lambda}R^{(0)}{}_{\lambda|\gamma)de} \ - \ T^{(0)}{}_{ab}{}^{g}R^{(0)}{}_{|\gamma)gde} \ - \ 2\nabla_{a}R^{(0)}{}_{b|\gamma)de}]
\end{aligned}
\tag{122}
$$

Substituting (122) into (118) gives

$$+\frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{gab}]R^{(0)ab}{}_{de}-2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gab}R^{(0)}{}_{abde}$$

$$-\ i\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde}\ +\ \frac{i\gamma}{6}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\nabla_{a}R^{(0)}{}_{|\gamma)bde}+2H^{(0)}{}_{ab}{}^{g}R^{(0)}{}_{|\gamma)gde}]$$

$$-\ \frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{pqr}]+\ 2i\gamma\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}\Pi'{}_{gde}]=0 \qquad (123)$$

Now consider the sigma five terms in (123). Using our new result result (115) allows for solving these terms, and we obtain

$$+\ \frac{\gamma}{6}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\nabla_{a}R^{(0)}{}_{|\gamma)bde}+2H^{(0)}{}_{ab}{}^{g}R^{(0)}_{|\gamma)gde}]$$

$$=\ \frac{\gamma}{6}\sigma^{pqrab}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[\sigma_{[d|g)\phi}\{\nabla_{a}T^{(0)}{}_{b|e]}{}^{f}+2H^{(0)}{}_{ab}{}^{c}T^{(0)}{}_{c|e]}{}^{\phi}\}]$$

$$=\ \frac{\gamma}{3}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqra}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}[\{\nabla_{[d|}T^{(0)}{}_{a|e]}{}^{f}+2H^{(0)}{}_{[d|a}{}^{c}T^{(0)}{}_{c|e]}{}^{\phi}\}] \qquad (124)$$

Hence we obtain

$$i\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde}\ =+\frac{i\gamma}{2}\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{gab}]R^{(0)ab}{}_{de}-2i\gamma\sigma^{g}{}_{(\alpha\beta|}R^{(1)}{}_{|\gamma)gab}R^{(0)}{}_{abde}$$

$$+\ \frac{\gamma}{3}\sigma^{g}{}_{(\alpha\beta|}\sigma^{pqra}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}[\{\nabla_{[d|}T^{(0)}{}_{a|e]}{}^{\phi}+2H^{(0)}_{[d|a}{}^{c}T^{(0)}{}_{|c|e]}{}^{\phi}\}]$$

$$-\ \frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{pqr}]+\ 2i\gamma\sigma^{g}{}_{(\alpha\beta|}[\nabla_{|\gamma)}\Pi'{}_{gde}]=0 \qquad (125)$$

We now look at the remaining unsolved term $-\frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{pqr}]$.

This term looks as though it will pose a serious problem. This term cannot be manipulated into a solvable term because of the placement of the free indices. Using the results found in [4] we have

$$- \frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}[\nabla_{|\gamma)}A^{(1)}{}_{pqr}] =$$

$$+\frac{\gamma}{(12)(24)}\sigma^{pqr}{}_{de(\beta\gamma|}\sigma_{pqr\epsilon\tau}T^{(0)\epsilon}{}_{kl}\sigma^{mns\phi\tau}[H^{(0)kl}{}_{g}\sigma^{g}{}_{|\alpha)\phi}A^{(1)}{}_{mns}$$

$$-\sigma_{k|\alpha)\phi}(\nabla_l A^{(1)}{}_{mns})] \tag{126}$$

Using the sigma matrix identities as given in ref. [3], it can be shown that these two terms cannot be written in the solvable form, that is with the same structure as $\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde}$. Hence we look at the origin of these terms. For the derivative of $T_{kl}{}^{\tau}$ we have the following Bianchi identity.

$$\nabla_\gamma T_{kl}{}^{\tau} = T_{\gamma[k|}{}^{\lambda}T_{\lambda|l]}{}^{\tau} + T_{\gamma[k}{}^{g}T_{g|l]}{}^{\tau} + T_{kl}{}^{\lambda}T_{\lambda\gamma}{}^{\tau} + T_{kl}{}^{g}T_{g\gamma}{}^{\tau} - \nabla_{[k|}T_{|l]\gamma}{}^{\tau} - R_{kl\gamma}{}^{\tau} \tag{127}$$

At first order this reduces to

$$\nabla_\gamma T_{kl}{}^{\tau Order(1)} = T^{(1)}{}_{\gamma[k|}{}^{\lambda}T^{(0)}{}_{\lambda|l]}{}^{\tau} - \nabla_{[k|}T_{|l]\gamma}{}^{\tau Order(1)} - R^{(1)}{}_{kl\gamma}{}^{\tau} \tag{128}$$

In references [2] and [4] it appears that $R^{(1)}{}_{kl\gamma}{}^{\tau}$ was set to zero. With the form of the curvature $R_{\alpha\beta de}$ and this choice of super current supertensor A_{abc} we will always be led to the term $\frac{i}{24}\sigma^{pqr}{}_{de(\alpha\beta|}\nabla_{|\gamma)}A^{(1)}{}_{pqr}{}^{(order2)}$ because of the the spinor derivative in the Bianchi identity (107) as given in (112). This term is not reducible as we require so it must be incorporated into this curvature. Hence we identify the following curvature at first order.

$$R^{(1)}{}_{kl\gamma}{}^{\tau} = \frac{1}{48}[2H^{(0)}{}_{klg}\sigma^{g}{}_{\gamma\lambda}\sigma^{pqr\lambda\tau}A^{(1)}{}_{pqr} - \sigma_{[k|\gamma\lambda}\sigma^{pqr\lambda\tau}(\nabla_{|l]}A^{(1)}{}_{pqr})] \tag{129}$$

This result was not arrived at in [2] and [4] and will have consequences for the application of even the first order results.

The second order form of this curvature is already solved in the Bianchi identity (127). All the quantities in this Bianchi identity are known. Hence it can be written in full in a later review. It is the role of this paper simply to arrive at the second order solution, and to overcome obstacles to obtaining this solution.

11 The Super-Current

At first the author believed that finding the supercurrent tensor ${A^{(2)}}_{abc}$ would result in closing the curvature identity (107). However that was before equation (115) was constructed and the before the significance of the (128) was realized. Thence finding the supercurrent ${A^{(2)}}_{abc}$ is nothing more than applying a condition available from the conventional constraints. The starting point in references [2] and [4] were the conventional constraints as listed in [2]. Among these constraints we have

$$ {T_{\alpha b}}^{\delta} = \frac{1}{48}\sigma_{b\alpha\lambda}\sigma^{pqr\lambda\delta}A_{pqr} \tag{130} $$

The choice of

$$ A_{pqr} = -i\gamma\sigma_{pqr\epsilon\tau}{T_{kp}}^{\epsilon}T^{kp\tau} \tag{131} $$

was made for on shell conditions, [2]. However this is a conventional constraint and therefore it can be imposed to all orders. Hence we can use this result. We have found $T_{\alpha b}{}^{\delta}$ and it is given in in equation (105). Hence we can solve for $A^{(2)}{}_{pqr}$. No modification to this super-current was required to close the identities other than this. Hence we use a suitable inverting operator along with our results (105) and (80) to obtain after som calculation

$$A^{(2)}{}_{gef} = -\frac{1}{20}\sigma_{gef\gamma\lambda}\sigma^{b\lambda\phi}T^{(2)}{}_{\phi b}{}^{\gamma} \tag{132}$$

.

$$= \frac{\gamma}{20}\sigma_{gef\gamma\lambda}\sigma^{b\lambda\phi}[\nabla_{\phi}(\frac{1}{4}A^{(1)}{}_{bmn} - 2L^{(1)}{}_{bmn})]T^{(0)mn\gamma} \tag{133}$$

12 Conclusions

We have solved the non-minimal case of string corrected supergravity, for D=10, N=1. This theory is believed to be the low energy realization of string theory. We found a procedure for solving the Bianchi identities to this order and thus maintained manifest supersymmetry to that order. Our solution required the intricate derivation of equation (39), which we used in conjunction with several other results and observations. In particular we had to form an Ansatz for the so called X tensor which would be consistent within all sectors of the Bianchi identities. The Ansatz that we found achieved this result.

Hence we found a mechanism which allows for closure of the H sector Bianchi identities and solved a problem that had existed for many years. We also solved the Bianchi identities in the torsion and curvature sectors at each dimension, showing consistency of our set of results. To achieve this in full, progress was held up as it was necessary to derive yet another identity, (115), which facilitated the elimination of otherwise unsolvable terms. Furthermore we also had to observe that, in contradiction to the results reported in ref. [2] and [4], we had to show that the curvature ${R^{(1)}}_{ab\gamma}{}^{\delta}$ was not in fact zero. These observations were not at all immediately transparent. We found ${R^{(1)}}_{ab\gamma}{}^{\delta}$ to be given by equation (133). We noted that the second order contribution ${R^{(2)}}_{ab\gamma}{}^{\delta}$, can be found already form the Bianchi identity (127) by direct substitution of our already found results at second order. It would simply be a long expression.

We saw how the X tensor was necessary for achieving consistent closure of the Bianchi identities and our candidate for this tensor, which succeeds in doing this, did not require the contribution of a second possible part, Y_{pqrdef}. Such a contribution would appear in fact to result in failure to close in the H sector. Hence we feel that out choice is in fact the unique result.

With ${H^{(2)}}_{\alpha\beta\delta}$ set to zero we obtained

$$
{H^{(2)}}_{d\alpha\beta} \;=\; {\sigma_{\alpha\beta}}^{g}[8i\gamma {H^{(0)}}_{def}{L^{(1)}}_{g}{}^{ef} \;-\; i\gamma {H^{(0)}}_{def}{A^{(1)}}_{g}{}^{ef}] \;-\; \frac{i\gamma}{12}{\sigma^{pqref}}_{\alpha\beta}[{H^{(0)}}_{def}{A^{(1)}}_{pqr}] \tag{134}
$$

${H^{(2)}}_{\alpha bg}$ must be extracted from (66). We use the following operator, $\hat{O}$, to obtain the symmetrized ${H^{(2)}}_{\alpha bg}$:

$$\hat{O} = [\frac{1}{2}\delta_{[a}{}^{d}\delta_{b]}{}^{g}\delta_{\alpha}{}^{\beta} \; - \; \frac{1}{12}\eta^{dg}\sigma_{ab\alpha}{}^{\beta} \; + \frac{1}{24}\delta_{[a}{}^{(d}\sigma_{b]}{}^{g)\beta}_{\alpha}] \tag{135}$$

After a very long calculation we obtain the result

$$\begin{aligned} H^{(2)}{}_{\alpha ab} = \; & 2\gamma[\nabla_{\alpha}(H^{(0)}_{[a|ef}H^{(0)}{}_{|b]}{}^{ef}) \; - \; \sigma_{ab\alpha}{}^{\phi}\nabla_{\phi}(H^{(0)}{}_{gef}H^{gef})] \\ & + 2i\gamma\sigma_{[a|\alpha\phi}T_{ef}{}^{\phi}\Pi^{(1)}{}_{|b]}{}^{ef} - \; 2i\gamma\sigma_{ab\alpha}{}^{\lambda}\sigma_{g\lambda\phi}T_{ef}{}^{\phi}\Pi^{(1)gef} \\ & - \; \frac{\gamma}{6}\sigma^{g}{}_{[a|\alpha}{}^{\phi}\sigma_{|b]\lambda\phi}T_{ef}{}^{\lambda}\Pi^{(1)}{}_{g}{}^{ef} - \; \frac{\gamma}{6}\sigma^{g}{}_{[a|\alpha}{}^{\phi}\sigma_{g\lambda\phi}T_{ef}{}^{\lambda}\Pi^{(1)}{}_{|b]}{}^{ef} \\ & - \; 4\gamma R^{(1)}{}_{\alpha[a|}{}^{ef}H^{(0)}{}_{|b]ef} + T^{(2)}{}_{\alpha ab} \end{aligned} \tag{136}$$

where $T^{(2)}{}_{\alpha ab}$ is given in equation (142). In the case of $H^{(2)}{}_{abc}$, the Bianchi identity has already given us the result. From the term $T_{\alpha\beta}{}^{E}H_{Ecd}$ in equation (5) we isolate an expression of the form

$$T^{(0)}{}_{\alpha\beta}{}^{g}H^{(2)}{}_{gcd} \; = \; i\sigma_{\alpha\beta}{}^{g}H^{(2)}{}_{gcd} \; = \; M_{\alpha\beta cd} \tag{137}$$

The right hand side contains now known torsions and curvatures. However they need only be substituted into (137) generating a long expression. We then use the fact that

$$\sigma_{a\alpha\beta}\sigma^{b\alpha\beta} \; = \; - \; 16\delta^{b}{}_{a} \tag{138}$$

and solve for $H^{(2)}{}_{gcd}$.

We obtained the full set or torsions and and curvatures,

$$T^{(2)}{}_{\alpha\beta}{}^{\lambda} = -\frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}T_{ef}{}^{\lambda} \tag{139}$$

$$T^{(2)}{}_{\alpha\beta}{}^{d} = -\frac{i\gamma}{6}\sigma^{pqref}{}_{\alpha\beta}H^{(0)d}{}_{ef}A^{(1)}{}_{pqr} \tag{140}$$

$$T^{(2)}{}_{\gamma g}{}^{\delta} = -\frac{\gamma}{2}[\nabla_{\gamma}A^{(1)}{}_{gef}]T^{(0)}{}_{ef}{}^{\delta} + 2R^{(1)}{}_{\gamma gef}T^{(0)}{}_{ef}{}^{\delta} \tag{141}$$

Extracting the symmetrized torsion $T^{(2)}{}_{\gamma gd}$ from (78) gives

$$\begin{aligned}
T^{(2)}{}_{\gamma ab} = &-\frac{\gamma}{2}[\nabla_{\gamma}A^{(1)}{}_{[a|ef}]H^{(0)}{}_{|b]}{}^{ef} + 2\gamma R^{(1)}{}_{\gamma[a|ef}H^{(0)}{}_{|b]}{}^{ef} - \frac{i\gamma}{12}\sigma^{pqrg}{}_{[a|\gamma\lambda}T^{(0)}{}_{|b]g}{}^{\lambda}A^{(1)}{}_{pqr} \\
&+\sigma_{ab\ \gamma}{}^{\phi}[+\frac{\gamma}{12}(\nabla_{\phi}A^{(1)}{}_{gef})H^{(0)gef} + \frac{\gamma}{3}R^{(1)}{}_{\phi\ gef}H^{(0)gef} - \frac{i\gamma}{72}\sigma^{pqreg}{}_{\phi\lambda}A^{(1)}{}_{pqr}T^{(0)}{}_{eg}{}^{\lambda}] \\
&+\sigma_{[a|}{}^{g}{}_{\gamma}{}^{\phi}[-\frac{\gamma}{2}(\nabla_{\phi}A^{(1)}{}_{|b]ef})H^{(0)}{}_{g}{}^{ef} - \frac{\gamma}{2}(\nabla_{\phi}A^{(1)}{}_{gef})H^{(0)}{}_{|b]}{}^{ef} \\
&-\frac{\gamma}{6}R^{(1)}{}_{\phi\ |b]ef}H^{(0)}{}_{g}{}^{ef} - \frac{\gamma}{6}R^{(1)}{}_{\phi\ gef}H^{(0)}{}_{|b]}{}^{ef} \\
&+\frac{i\gamma}{144}A^{(1)}{}_{pqr}[\sigma^{pqre}{}_{|b]\phi\lambda}T^{(0)}{}_{eg}{}^{\lambda} + \sigma^{pqre}{}_{g\ \phi\lambda}T^{(0)}{}_{e|b]}{}^{\lambda}]]
\end{aligned} \tag{142}$$

where

$$\nabla_{\gamma}A^{(1)}{}_{gef} = i\gamma\sigma_{gef\ \epsilon\tau}T^{(0)}{}_{kp}{}^{\epsilon}[2T^{(0)kp\lambda}T^{(0)}{}_{\gamma\lambda}{}^{\tau} - \frac{1}{2}\sigma^{mn}{}_{\gamma}{}^{\tau}R^{(0)kp}{}_{mn}] \tag{143}$$

$$R^{(2)}{}_{\alpha\beta de} = -\frac{i\gamma}{12}\sigma^{pqref}{}_{\alpha\beta}A^{(1)}{}_{pqr}R_{efde} - 2i\sigma^{g}{}_{\alpha\beta}[L^{(2)}{}_{gde} + \frac{1}{4}A^{(2)}{}_{gde}] \tag{144}$$

We also find the adjusted curvature $R_{kl\gamma}{}^{\tau}$. For $R^{(2)}{}_{\alpha\beta de}$ we have reduced it to solvable form. After imposing conditions (133) and (115) on (110) we obtain

$$\begin{aligned} i\sigma^{g}{}_{(\alpha\beta|}R^{(2)}{}_{|\gamma)gde} \; &= \sigma^{g}{}_{(\alpha\beta|}[\frac{i\gamma}{2}[\nabla_{|\gamma)}A^{(1)}{}_{gab}]R^{(0)ab}{}_{de} - 2i\gamma R^{(1)}{}_{|\gamma)gab}R^{(0)}{}_{abde} \\ + \frac{\gamma}{3}\sigma^{pqra}{}_{g|\gamma)\phi}A^{(1)}{}_{pqr}\{\nabla_{[d|}T^{(0)}{}_{a|e]}{}^{\phi} &+ 2H^{(0)}_{[d|a}{}^{c}T^{(0)}{}_{c|e]}{}^{\phi}\} + \; 2i\gamma\nabla_{|\gamma)}\Pi'{}_{gde}] = 0 \end{aligned} \tag{145}$$

$R^{(2)}{}_{\alpha\beta de}$ can be extracted from the above result.

Finally we have found the supercurrent $A^{(2)}{}_{abc}$

$$= \frac{\gamma}{20}\sigma_{gef\gamma\lambda}\sigma^{b\lambda\phi}[\nabla_{\phi}(\frac{1}{4}A^{(1)}{}_{bmn} - 2L^{(1)}{}_{bmn})]T^{(0)mn\gamma} \tag{146}$$

13 Appendix I: Conventions

It has been shown that the Lorentz Chern-Simmons Form, Q_{ABC}, can be defined in superspace for the various dimensionalities as follows, [1],

$$\begin{aligned} Q_{\alpha\beta\gamma} &= \frac{1}{2}\omega_{(\alpha|}{}^{ef}R_{|\beta\gamma)gf} - \frac{1}{3}\omega_{(\alpha|}{}^{ef}\omega_{|\beta|}{}^{g}{}_{e}\omega_{|\gamma)ef} \\ Q_{\alpha\beta c} &= \omega_{(\alpha|}{}^{ef}R_{|\beta)cef} + \omega_{c}{}^{ef}R_{\alpha\beta cef} - \omega_{(\alpha|}{}^{ef}\omega_{|\beta)}{}^{g}{}_{e}\omega_{cgf} \\ Q_{\alpha bc} &= \omega_{(\alpha|}{}^{ef}R_{bcef} + \omega_{c}{}^{ef}R_{|c]\alpha ef} - \omega_{[a|}{}^{ef}\omega_{|c]}{}^{g}{}_{e}\omega_{\alpha gf} \\ Q_{abc} &= \frac{1}{2}\omega_{[a|}{}^{ef}R_{|bc]ef} - \frac{1}{3}\omega_{[a|}{}^{ef}\omega_{|b|}{}^{g}{}_{e}\omega_{|c]gf} \end{aligned} \tag{147}$$

D=10, N=1 appears in many formalisms, possibly related by a Weyl Transformation,[1]. For a review of this theory at zeroth order see [3]. Hence we require from D=10, N=1 Supergravity the field strength G_{ABC}, (see references [1], and many references therein).

$$G_{ABC} = \frac{1}{2}\partial_{[B|}B_{|BC)} \tag{148}$$

or it can be coupled to a Yang Mills Supermultiplet, [1]

In this paper we work with the modified field strength, H_{ABC}, as defined in equation equation (13) as opposed to G_{ABC} in reference [2]. We also have the supercurrent supertensor A_{ABC} as defined in equation (24) for on shell conditions, and later modified to second order in equation (133). This modification includes the string tension parameter α. This correction was first given in [1].

In this paper we use γ where.

$$\gamma = const.\alpha \tag{149}$$

The field strength L_{ABC} is simply given by

$$H_{ABC} = -2L_{ABC} \tag{150}$$

.

We have the curvatures and torsions $T_{AB}{}^{C}$, and curvatures, R_{ABCD}, for their various dimensions We also encounter the super field fields ϕ and χ which is given by

$$\chi_\alpha = -\frac{1}{2}\nabla_\alpha \phi \tag{151}$$

13.1 The Sigma Matrix Algebra

A considerable array of sigma matrix identities can be found in ref. [3]. For this work we require to know only those listed here. Details of all conventions are given in [3]. To make our work self contained however we start form first principles as follows. Spinors in D=10 space-time dimensions are sixteen component objects. Let dotted indices be right handed components and un-dotted indices left handed. Let let space-time indices be in small Roman script and spinor indices be Greek letters. We deal only with purely left handed (chiral supergravity) spinors so no dotted indices appear.

The local ten dimensional metric η_{ab} has signature $[+,-,-....]$. The sigma matrices are therefore 16 by 16 matrices which satisfy the usual Dirac algebra with anti-commutator as follows

$$\sigma_{a\alpha\beta}{\sigma_b}^{\beta\gamma} + \sigma_{b\alpha\beta}{\sigma_a}^{\beta\gamma} = -2\eta_{ab}{\sigma_\alpha}^\gamma \tag{152}$$

Hence we have

$$\sigma_{a\alpha\beta}{\sigma_b}^{\beta\gamma} = -\sigma_{b\alpha\beta}{\sigma_a}^{\beta\gamma} - 2\eta_{ab}{\sigma_\alpha}^\gamma \tag{153}$$

Or we have

$$\sigma_{a\alpha\beta}{\sigma_b}^{\beta\gamma} = -{\sigma_{ab\alpha}}^\gamma - \eta_{ab}{\sigma_\alpha}^\gamma \tag{154}$$

This defines the object $\sigma_{ab\alpha}{}^{\gamma}$. Similarly

$$\sigma_{a\alpha\beta}\sigma_{bc\gamma}{}^{\beta} = -\sigma_{abc\alpha\gamma} - \frac{1}{2!}\eta_{a[b|}\sigma_{|c]\alpha\gamma} \tag{155}$$

And so on as we can build up related products. Hence all the sigma matrices are anti-symmetric in their vector indices. Sigma matrices with odd numbers of vector indices are anti-symmetric in their spinor indices. Symmetrization and anti-symmetrization is defined as follows.

$$A_{(a|}A_{|b)} = A_a A_b + A_b A_a \tag{156}$$

$$A_{[a|}A_{|b]} = A_a A_b - A_b A_a \tag{157}$$

The only identities from [3] which we use in this paper are listed below. The remaining ones that we use are derived in the appendix. We have the following identities which we used once or more

$$\sigma_{a\alpha\beta}\sigma^{b\alpha\beta} = -\delta_a{}^b \tag{158}$$

$$\sigma_{ab\alpha}{}^{\beta}\sigma^{cd}{}_{\beta}{}^{\alpha} = -\delta_{[a|}{}^{c}\delta_{|b]}{}^{d} \tag{159}$$

$$\sigma_{abc\alpha\beta}\sigma^{def\alpha\beta} = -\delta_{[a|}{}^{d}\delta_{b}{}^{e}\delta_{|c]}{}^{f} \tag{160}$$

$$\sigma_{abc\alpha\beta}\sigma^{abc\gamma\delta} = -8.3!.\delta_{[\alpha|}{}^{\gamma}\delta_{|\beta]}{}^{\delta} \tag{161}$$

$$\sigma_{abc\alpha\beta}\sigma^{abc}{}_{\gamma\delta} = -2.3!.\delta^{a}{}_{\alpha[\gamma|}\sigma_{a|\delta]\beta} \tag{162}$$

$$\sigma_{a\alpha\beta}\sigma^{a\beta\gamma} = -10\delta_{\alpha}^{\gamma} \tag{163}$$

Finally we have the important result

$$\sigma^{pqr}{}_{ef\alpha\beta} = [\eta^{[p}{}_{e}\eta_{f}^{q}\sigma^{r]}{}_{\alpha\beta} - \frac{1}{2}\sigma_{ef\alpha}{}^{\phi}\sigma^{pqr}{}_{\phi\beta}] \tag{164}$$

To complete the set we require also equation (40).

14 Appendix II

Using the notation of [6] we define torsions and curvatures as follows,

$$[\nabla_A, \nabla_B\} = T_{AB}{}^{C} + \frac{1}{2}R_{ABd}{}^{e}M_{e}{}^{d} \tag{165}$$

In this paper we required the following,

$$T_{(\alpha\beta|}{}^{\lambda}T_{|\gamma)\lambda}{}^{\delta} \;-\; T_{(\alpha\beta|}{}^{g}T_{|\gamma)g}{}^{\delta} \;-\; \nabla_{(\alpha|}T_{|\beta\gamma)}{}^{\delta} - \frac{1}{4}R^{(2)}{}_{(\alpha\beta|de}\sigma^{de}{}_{|\gamma)}{}^{\delta} \;=\; 0 \tag{166}$$

$$T_{(\alpha\beta|}{}^{\lambda}T_{|\gamma)\lambda}{}^{d} \;-\; T_{(\alpha\beta|}{}^{g}T_{|\gamma)g}{}^{d} \;-\; \nabla_{(\alpha|}T_{\beta\gamma)}{}^{d} \;=\; 0 \tag{167}$$

$$T_{(\alpha\beta|}{}^{\lambda}R_{|\gamma)\lambda de} \;-\; T_{(\alpha\beta|}{}^{g}R_{|\gamma)gde} \;-\; \nabla_{(\alpha|}R_{\beta\gamma)de} \;=\; 0 \tag{168}$$

$$\nabla_{\alpha}R_{abde} \;=\; T_{\alpha[a|}{}^{E}R_{E|b]de} \;+\; T_{ab}{}^{E}R_{E\alpha de} \;-\; \nabla_{[a|}R_{|b]\alpha de} \tag{169}$$

15 Appendix III

A lengthy derivation available from author.

16 Appendix IV: Calculation of Equation (37)

A lengthy derivation available from author.

17 Appendix V: Calculation of Equation (62)

We have the term

$$\frac{+i\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[R^{(0)}{}_{|\gamma)def} \ - \ 2\nabla_{|\gamma)}H^{(0)}{}_{def}] \tag{170}$$

$$= \ \frac{+i\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[-i\sigma_{[e|\gamma)\phi}T^{(0)\phi}{}_{d|f]} \ - \ 2\frac{i}{4}\sigma_{[d|\gamma)\phi}T^{(0)\phi}{}_{|ef]}] \tag{171}$$

$$= \ (-i)\frac{+i\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[2\sigma_{e|\gamma)\phi}T^{(0)\phi}{}_{df} \ + \ \frac{1}{2}[2\sigma_{d|\gamma)\phi}T^{(0)\phi}{}_{ef} \ - \ 4\sigma_{e\gamma)\phi}T^{(0)\phi}{}_{df}]] \tag{172}$$

$$= \ \frac{+\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}\sigma_{d|\gamma)\phi}T^{(0)\phi}{}_{\varepsilon f} \tag{173}$$

Consider

$$\sigma^{pqref}{}_{\alpha\beta}\sigma_{d\gamma\phi}T^{(0)}{}_{ef}{}^{\phi}A^{(1)}{}_{pqr} \quad = [\eta_e^{[p}\eta_f^{q}\sigma_{\alpha\beta}^{r]} \ - \ \frac{1}{2}\sigma_{ef(\alpha|}{}^{\phi}\sigma^{pqr}{}_{|\beta)\phi}]A^{(1)}{}_{pqr}\sigma_{d\gamma\lambda}T^{(0)ef\lambda} \tag{174}$$

$$= [6\sigma_{\alpha\beta}^{g}A^{(1)}{}_{gef}\sigma_{d\gamma\lambda}T^{(0)ef\lambda} \ - \ \frac{1}{2}\sigma_{ef(\alpha|}{}^{\phi}\sigma^{pqr}{}_{|\beta)\phi}]A^{(1)}{}_{pqr}\sigma_{d\gamma\lambda}T^{(0)ef\lambda}$$

$$= \lambda_1 \ + \ \lambda_2 \tag{175}$$

Using the definition of $A^{(1)}{}_{gef}$ gives

$$\lambda_2 \ = \ - \ \frac{1}{2}\sigma_{ef(\alpha|}{}^{\phi}\sigma^{pqr}{}_{|\beta)\phi}[-i\gamma\sigma_{pqr\epsilon\tau}]\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau} \tag{176}$$

Using ref. [3] and the appropriate sigma matrix result, gives

$$= \ + \ \frac{1}{2}\sigma_{ef(\alpha|}{}^{\phi}[-12i\sigma^{g}{}_{|\beta)[\epsilon|}\sigma_{g|\tau]\phi}]\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau} \tag{177}$$

With the anti symmetry in ϵ and τ gives a factor of 2,

$$= \ (-12i)\gamma\sigma_{ef(\alpha|}{}^{\phi}\sigma^{g}{}_{|\beta)\epsilon}\sigma_{g\tau\phi}\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau} \tag{178}$$

We also have the basic result that

$$\sigma_{ef\alpha}{}^{\phi}\sigma_{g\tau\phi} \ = \ - \ \eta_{g[e}\sigma_{f]\alpha\tau} \ - \ \sigma_{gef\alpha\tau} \tag{179}$$

Similarly,

$$\sigma_{ef\tau}{}^{\phi}\sigma_{g\alpha\phi} \ = \ - \ \eta_{g[e}\sigma_{f]\alpha\tau} \ - \ \sigma_{gef\tau\alpha} \tag{180}$$

Now add the above two equations and use the anti-symmetry in the spinor indices of $\sigma_{gef\tau\alpha}$ to get

$$\sigma_{ef\alpha}{}^{\phi}\sigma_{g\tau\phi} \ = \ - \ 2\eta_{g[e}\sigma_{f]\alpha\tau} \ - \ \sigma_{ef\tau}{}^{\phi}\sigma_{g\alpha\phi} \tag{181}$$

Hence λ_2 becomes

$$\lambda_2 \ = \ (-12i)\gamma\sigma^{g}{}_{(\beta|\epsilon}[-2\eta_{g[e|}\sigma_{|f]|\alpha)\tau} \ - \ \sigma_{ef\tau}{}^{\phi}\sigma_{g|\alpha)\phi}]\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau} \tag{182}$$

Noting also the antisymmetry in e and f, we get

$$= \ +48i\gamma\eta_{ge}\sigma_{f(\alpha|\tau}\sigma^{g}{}_{|\beta)\epsilon}\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau}$$

$$+\ 12i\gamma\sigma_{ef\tau}{}^{\phi}\sigma_{g(\alpha|\phi}\sigma^{g}{}_{|\beta)\epsilon}\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau} \tag{183}$$

We have the result,

$$\sigma_{g(\alpha|\phi}\sigma^{g}{}_{|\beta)\epsilon} \ = \ - \ \sigma_{g\alpha\beta}\sigma^{g}{}_{\epsilon\phi} \tag{184}$$

We can also show that can $[\sigma^{g}{}_{[\epsilon|\phi}\sigma_{ef|\tau]\phi}] \quad = \quad -2\sigma_{gef\epsilon\tau}$

Hence

$$\lambda_2 = \ +\ 48i\gamma\sigma_{e(\alpha|\epsilon}\sigma_{f|\beta)\tau}\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau}$$

$$-\ 12i\gamma\sigma_{g\alpha\beta}[\frac{1}{2}\sigma^{g}{}_{[\epsilon|\phi}\sigma_{ef|\tau]\phi}\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau} \tag{185}$$

$$= \ +\ 48i\gamma\sigma_{e(\alpha|\epsilon}\sigma_{f|\beta)\tau}\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau}$$

$$+\ 12i\gamma\sigma^{g}_{\alpha\beta}\sigma_{gef\epsilon\tau}\sigma_{d|\gamma)\lambda}T^{(0)ef\lambda} \tag{186}$$

So using this and also the definition of $A^{(1)}{}_{gef}$ we finally get

$$= \ \ +\ 48\ (i)\gamma\sigma_{e(\alpha|\epsilon}\sigma_{f|\beta)\tau}\sigma_{d|\gamma)\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau}$$

$$-12\sigma^{g}_{\alpha\beta}A^{(1)}{}_{gef}\sigma_{d|\gamma\lambda}T^{(0)ef\lambda} \tag{187}$$

This was the second term in equation (117). Adding to λ_1 gives

$$\lambda_1 \; + \; \lambda_2 \; = \; + \; 48(i)\gamma\sigma_{e(\alpha|\epsilon}\sigma_{f|\beta)\tau}\sigma_{d\gamma\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau}$$
$$+ \; (6 \; - \; 12)\sigma^{g}_{\alpha\beta}A^{(1)}{}_{gef}\sigma_{d\gamma\lambda}T^{(0)ef\lambda} \tag{188}$$

Hence introducing the symmetries over α, β, γ we get the final result,

$$\frac{+i\gamma}{24}\sigma^{pqref}{}_{(\alpha\beta|}A^{(1)}{}_{pqr}[R^{(0)}{}_{|\gamma)def} \; - \; 2\nabla_{|\gamma)}H^{(0)}{}_{def}]$$
$$= \quad -\frac{\gamma}{4}\sigma^{g}_{(\alpha\beta|}A^{(1)}{}_{gef}\sigma_{d|\gamma)\lambda}T^{(0)ef\lambda}$$
$$+ \, 4i\gamma^2\sigma_{e(\alpha|\epsilon}\sigma_{f(\beta|\tau}\sigma_{d|\gamma)\lambda}T^{(0)ef\lambda}T^{(0)}{}_{kp}{}^{\epsilon}T^{(0)kp\tau} \tag{189}$$

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